

# ON GEOMETRIC ASPECTS OF QUATERNIONIC AND OCTONIONIC SLICE REGULAR FUNCTIONS

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**ABSTRACT.** The purpose of this paper is twofold. One is to enrich from a geometrical point of view the theory of octonionic slice regular functions. We first prove a boundary Schwarz lemma for slice regular self-mappings of the open unit ball of the octonionic space. As applications, we obtain two Landau-Toeplitz type theorems for slice regular functions with respect to regular diameter and slice diameter respectively, together with a Cauchy type estimate. Along with these results, we introduce some new and useful ideas, which also allow to prove the minimum principle and one version of the open mapping theorem. Another is to strengthen a version of boundary Schwarz lemma first proved in [37] for quaternionic slice regular functions, with a completely new approach. Our quaternionic boundary Schwarz lemma with optimal estimate improves considerably a well-known Osserman type estimate and provides additionally all the extremal functions.

## CONTENTS

1. Introduction	1
2. Preliminaries	6
2.1. Octonions	6
2.2. Octonionic slice regular functions	9
3. Proof of Theorem 1.1	13
4. Proofs of Theorems 1.4, 1.5 and 1.6	19
5. Geometric properties of octonionic slice regular functions	24
5.1. The minimum principle and the open mapping theorem	24
5.2. The growth, distortion and covering theorems	29
6. A new and sharp boundary Schwarz lemma for quaternionic slice regular functions	30
6.1. Quaternionic slice regular functions	30
6.2. Formulation and proof of quaternionic boundary Schwarz lemma	32
6.3. Some corollaries of Theorem 6.6	38
References	40

## 1. INTRODUCTION

A promising theory of quaternion-valued functions of one quaternionic variable, now called slice regular functions, was initially introduced by Gentili and Struppa in [15, 16] and has been extensively studied over the past few years. It turns out to be significantly different from the more classical theory of regular (or monogenic) functions in

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the sense of Cauchy-Fueter and has elegant applications to the functional calculus for noncommutative operators [8], Schur analysis [2] and the construction and classification of orthogonal complex structures on dense open subsets of  $\mathbb{R}^4$  [11]. Meanwhile, the theory of quaternionic slice regular functions has been extended to octonions in [17]. The related theory of slice monogenic functions on domains in the paravector space  $\mathbb{R}^{n+1}$  with values in the Clifford algebra  $\mathbb{R}_n$  was introduced in [7, 9]. For the detailed up-to-date theory, we refer the reader to the monographs [8, 14] and the references therein. More recently, a connection between slice monogenic and monogenic functions was investigated in [6] by means of Radon and dual Radon transforms. These function theories were also unified and generalized in [19] by means of the concept of slice functions on the so-called quadratic cone of a real alternative  $*$ -algebra, based on a slight modification of a well-known construction due to Fueter. The theory of slice regular functions on real alternative  $*$ -algebras is by now well-developed through a series of papers [21–25] mainly due to Ghiloni and Perotti after their seminal work [19].

Exactly as the quaternions  $\mathbb{H}$  being the only real *associative normed division* algebra of dimension greater than 2, the theory of quaternionic slice regular functions should be the most beautiful one among these function theories mentioned above. This is indeed the case. Such a special class of functions enjoys many nice properties similar to those of classical holomorphic functions of one complex variable. Among them, we particularly mention the open mapping theorem, which is by now known to hold only for slice regular functions on symmetric slice domains in  $\mathbb{H}$  and allows us to prove the Koebe type one-quarter theorem for slice regular extensions to the quaternionic ball of univalent holomorphic functions on the unit disc of the complex plane, see Theorem [36, Theorem 4.9] for details. Furthermore, from the analytical perspective, only for quaternionic slice regular functions, the regular product and regular quotient have an intimate connection with the usual pointwise product and quotient. It is exactly this connection which plays a crucial role in many arguments, see the monograph [14] and the recent paper [37] for more details.

Now a rather natural question arises of whether the nice properties enjoyed by quaternionic slice regular functions can be proved for slice regular functions on domains over the octonions  $\mathbb{O}$ , the only *normed division* algebra among alternative algebras over  $\mathbb{R}$  of dimension greater than 4. In this paper, we introduce some new and useful ideas to overcome difficulties brought by the non-commutativity and the non-associativity and in turn to show that to great extent, this is indeed the case.

First of all, we are going to focus on the boundary behavior of octonionic slice regular functions, analogous to that of holomorphic functions. More specifically, we shall prove a boundary Schwarz lemma for slice regular self-mappings of the open unit ball  $\mathbb{B} := \{w \in \mathbb{O} : |w| < 1\}$ . To state its precise content, we first introduce some necessary notations. For a given element  $\xi = x + yI \in \mathbb{O}$  with  $I$  being an element of the unit 6-dimensional of purely imaginary octonions

$$(1.1) \quad \mathbb{S} = \{w \in \mathbb{O} : w^2 = -1\},$$

we denote by  $\mathbb{S}_\xi$  the associated 6-dimensional sphere (reduces to the point  $\xi$  when  $\xi$  is real):

$$\mathbb{S}_\xi := x + y\mathbb{S} = \{x + yJ : J \in \mathbb{S}\}.$$

It is well known that  $\mathbb{S}_\xi$  is exactly the conjugacy class of  $\xi$  (cf. [42, Proposition 2, Corollary 2.1]). For any three octonions  $u, v, w \in \mathbb{O}$ , the *Lie bracket* of  $u, v$  and the *associator* of  $u, v, w$  are respectively defined to be

$$[u, v] := uv - vu, \quad [u, v, w] := (uv)w - u(vw).$$

We also denote by  $\langle \cdot, \cdot \rangle$  the standard Euclidean inner product on  $\mathbb{O} \cong \mathbb{R}^8$ . Now our first main result can be stated as follows:

**Theorem 1.1.** *Let  $\xi \in \partial\mathbb{B}$  and  $f$  be a slice regular function on  $\mathbb{B} \cup \mathbb{S}_\xi$  such that  $f(\mathbb{B}) \subseteq \mathbb{B}$  and  $f(\xi) \in \partial\mathbb{B}$ . Then*

(i) *it holds that*

$$(1.2) \quad \begin{aligned} \frac{\partial|f|}{\partial\xi}(\xi) &= \bar{\xi} \left( f(\xi) \overline{f'(\xi)} + [\bar{\xi}, f(\xi) \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}] + 2[\xi, f(\xi), R_{\bar{\xi}} R_{\xi} f(\xi)] \right) \\ &\geq \frac{|1 - \langle f(0), f(\xi) \rangle|^2}{1 - |f(0)|^2}, \end{aligned}$$

where  $\frac{\partial|f|}{\partial\xi}(\xi)$  is the directional derivative of  $|f|$  along the direction  $\xi$  at the boundary point  $\xi \in \partial\mathbb{B}$ ;

(ii) *if further  $f(0) = 0$  and  $f(\xi) = \xi$ , then*

$$(1.3) \quad \frac{\partial|f|}{\partial\xi}(\xi) = f'(\xi) - [\xi, R_{\bar{\xi}} R_{\xi} f(\xi)] \geq \frac{2}{1 + \operatorname{Re} f'(0)}.$$

Moreover, equality holds for the inequality in (1.3) if and only if  $f$  is of the form

$$(1.4) \quad f(w) = w(1 - wa\bar{\xi})^{-*} * (w\bar{\xi} - a)$$

for some constant  $a \in [-1, 1)$ .

For the precise definitions of  $R_{\bar{\xi}} R_{\xi} f(\xi)$  and  $*$ -product appeared in Theorem 1.1, see (3.1), (3.2) and Sect. 2 below. It turns out that  $R_{\bar{\xi}} R_{\xi} f(\xi)$  is intimately related to the second coefficient in a new series expansion of slice regular function  $f$ , see [46, Theorem 4.1] for the quaternionic case and [22, Theorem 5.4] for the real alternative  $*$ -algebra case. Moreover, it is well worth remarking here that in contrast to the complex case, the Lie bracket in (1.3) does not necessarily vanish and  $f'(\xi)$  may not be a real number. An explicit counterexample can be found in Example 3.4 below.

Although the key ingredient in proving Theorem 1.1 is still a careful consideration of the geometrical information of  $f$  at its prescribed contact point  $\xi$  (i.e.  $f(\xi) \in \partial\mathbb{B}$ ), two crucial difficulties arise in the octonionic setting. One is that the case of  $\xi$  being a contact point of  $f$  (i.e.  $f(\xi) \in \partial\mathbb{B}$ ) can not be reduced to the case of  $\xi$  being a boundary fixed point of  $f$  (i.e.  $f(\xi) = \xi \in \partial\mathbb{B}$ ); the other is that, because of the lack of associativity in  $\mathbb{O}$ , there is in general no nice connection between the regular product and the usual pointwise product unlike in the quaternionic setting. The peculiarities of the non-associative setting produce a completely new phenomenon, called the *camshaft effect* in [20]: an isolated zero of a slice regular function  $f$  is not necessarily a zero for the regular product  $f * g$  of  $f$  with another slice regular function  $g$ . Therefore, the method used in our recent work [37] fails in the present setting to get some satisfactory and even sharp estimate. Fortunately, we can come up with an effective approach to overcome partly these technical difficulties mentioned above. In the quaternionic case, with a completely new approach, we can strengthen a result first proved in [37] by the author and Ren, analogous to Theorem 1.1. Our quaternionic boundary Schwarz lemma with optimal estimate involves a Lie bracket, improves considerably a well-known Osserman type estimate and provides additionally all the extremal functions; see Theorem 6.6 below for details.

Let  $f$  be as described in Theorem 1.1. Notice that the directional derivative  $\frac{\partial f}{\partial \xi}(\xi)$  of  $f$  along the direction  $\xi$  at the boundary point  $\xi \in \partial \mathbb{B}$  satisfies that

$$\frac{\partial f}{\partial \xi}(\xi) = \xi f'(\xi),$$

thus the obvious inequality

$$\left| \frac{\partial f}{\partial \xi}(\xi) \right| \geq \frac{\partial |f|}{\partial \xi}(\xi)$$

results in:

**Corollary 1.2.** *Let  $\xi \in \partial \mathbb{B}$  and  $f$  be a slice regular function on  $\mathbb{B} \cup \{\xi\}$  such that  $f(\mathbb{B}) \subseteq \mathbb{B}$ ,  $f(0) = 0$  and  $f(\xi) = \xi$ . Then*

$$|f'(\xi)| \geq \frac{2}{1 + \operatorname{Re} f'(0)}.$$

Moreover, equality holds for the last inequality if and only if  $f$  is of the form

$$f(w) = w(1 - wa\bar{\xi})^{-*} * (w\bar{\xi} - a)$$

for some constant  $a \in [-1, 1)$ .

We shall give some applications of Theorem 1.1 to the study of geometric properties and rigidity of slice regular functions. We first recall the notion of *regular diameter*, a suitable tool to measure the image of the open unit ball  $\mathbb{B}$  of the octonionic space  $\mathbb{O}$  through a slice regular function.

**Definition 1.3.** Let  $f$  be a slice regular function on  $\mathbb{B}$  with Taylor expansion

$$f(w) = \sum_{n=0}^{\infty} w^n a_n.$$

For each  $r \in (0, 1)$ , the *regular diameter* of the image of  $r\mathbb{B}$  under  $f$  is defined to be

$$(1.5) \quad \tilde{d}(f(r\mathbb{B})) := \max_{u, v \in \mathbb{B}} \max_{|w| \leq r} |f_u(w) - f_v(w)|,$$

where

$$f_u(w) := \sum_{n=0}^{\infty} w^n (u^n a_n), \quad f_v(w) := \sum_{n=0}^{\infty} w^n (v^n a_n).$$

The *regular diameter* of the image of  $\mathbb{B}$  under  $f$  is defined to be

$$(1.6) \quad \tilde{d}(f(\mathbb{B})) := \lim_{r \rightarrow 1^-} \tilde{d}(f(r\mathbb{B})).$$

As a first application of Theorem 1.1, we have the following Landau-Toeplitz type theorem for octonionic slice regular functions, whose quaternionic version was proved in [12].

**Theorem 1.4.** *Let  $f$  be a slice regular function on  $\mathbb{B}$  such that*

$$\tilde{d}(f(\mathbb{B})) = 2.$$

Then

$$(1.7) \quad \tilde{d}(f(r\mathbb{B})) \leq 2r$$

for each  $r \in (0, 1)$ , and

$$(1.8) \quad |f'(0)| \leq 1.$$

Moreover, equality holds in (1.7) for some  $r_0 \in (0, 1)$ , or in (1.8), if and only if  $f$  is an affine function

$$f(w) = f(0) + wf'(0).$$

Let  $E, \Omega$  be two subsets of  $\mathbb{O}$  and  $f : \Omega \rightarrow \mathbb{O}$  a function. We denote by  $\text{diam } E = \sup_{z, w \in E} |z - w|$  the Euclidean diameter of  $E$  and define the *slice diameter* of the image of  $\Omega$  under  $f$  to be

$$(1.9) \quad \widehat{d}(f(\Omega)) := \sup_{I \in \mathbb{S}} \text{diam } f(\Omega_I),$$

where  $\Omega_I$  denotes the intersection  $\Omega \cap \mathbb{C}_I$  of  $\Omega$  and  $\mathbb{C}_I$  the complex plane determined by  $I$ , and  $\mathbb{S}$  is the same as in (1.1). Thus we have another version of Landau-Toeplitz type theorem with respect to slice diameter.

**Theorem 1.5.** *Let  $f$  be a slice regular function on  $\mathbb{B}$  such that*

$$\widehat{d}(f(\mathbb{B})) = 2.$$

*Then*

$$(1.10) \quad \text{diam}(f(r\mathbb{B}_I)) \leq 2r$$

*for each  $r \in (0, 1)$  and each  $I \in \mathbb{S}$ , and*

$$(1.11) \quad |f'(0)| \leq 1.$$

*Moreover, equality holds in (1.10) for some  $r_0 \in (0, 1)$  and  $I_0 \in \mathbb{S}$ , or in (1.11), if and only if  $f$  is an affine function*

$$f(w) = f(0) + wf'(0).$$

As a second application of Theorem 1.1, we have the following Cauchy type estimate, which is an analogue of an old result due to Poukka (see [33]):

**Theorem 1.6.** *Let  $f$  be a bounded slice regular function on  $\mathbb{B}$  and  $d := \text{Diam } f(\mathbb{B})$  the Euclidean diameter of the image set  $f(\mathbb{B})$ . Then the inequality*

$$(1.12) \quad \frac{|f^{(n)}(0)|}{n!} \leq \frac{1}{2}d$$

*holds for every positive integer  $n \in \mathbb{N}$ . Moreover, equality holds in (1.12) for some  $n_0 \in \mathbb{N}$  if and only if*

$$f(w) = f(0) + \frac{1}{2}w^{n_0}de^{I\theta}$$

*for some  $I \in \mathbb{S}$  and some  $\theta \in \mathbb{R}$ .*

It is noteworthy here that inequality (1.12) easily follows from the classical result due to Poukka together with the splitting lemma for slice regular functions, or alternatively from Cauchy integral formula. The point here is to prove the last statement in the theorem.

Next we use some ideas developed in the proof of Theorem 1.1 to prove other properties of octonionic slice regular functions, among which are the minimum principle and the open mapping theorem.

**Theorem 1.7.** *Let  $f : \Omega \rightarrow \mathbb{O}$  be a slice regular function on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$ . If  $|f|$  attains a local minimum at some point  $w_0 \in \Omega \cap \mathbb{R}$ , then either  $f(w_0) = 0$  or  $f$  is constant.*

We shall give two proofs of the preceding theorem. Both of them involve a variational argument. The first one also provides a completely new and quite elementary approach to the maximum and the minimum principles for holomorphic functions of one complex variable. The second one is to reduce this theorem to the maximum principle (Theorem 4.3), based on a nice connection between the Euclidean norm of the slice regular function  $f$  and that of its regular reciprocal  $f^{-*}$  (Proposition 5.1). Furthermore, it seems that the restriction of  $w_0$  belonging to  $\Omega \cap \mathbb{R}$  in the preceding theorem is superfluous. There are some additional obstacles to prove the general case that  $w_0 \in \Omega \setminus \mathbb{R}$ ; see Remark 5.3 below for more details. If this restriction could be removed, the general minimum principle would immediately follow and in turn would imply the open mapping theorem analogous to [14, Theorem 7.7]. Here we can merely prove the following version of the open mapping theorem using a method different from that of [14, Theorem 7.4].

**Theorem 1.8.** *Let  $f : \Omega \rightarrow \mathbb{O}$  be a nonconstant slice regular function on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$ . If  $U$  is a symmetric open subset of  $\Omega$ , then  $f(U)$  is open. In particular,  $f(\Omega)$  is open.*

Theorem 1.8 is sufficient for proving an octonionic version of the classical Koebe one-quarter theorem for slice regular extensions to the octonionic ball  $\mathbb{B}$  of univalent holomorphic functions on the unit disc of the complex plane.

**Theorem 1.9.** *Let  $f$  be a slice regular function on  $\mathbb{B}$  such that its restriction  $f_I$  to  $\mathbb{B}_I$  is injective and  $f(\mathbb{B}_I) \subseteq \mathbb{C}_I$  for some  $I \in \mathbb{S}$ . If  $f(0) = 0$  and  $f'(0) = 1$ , then it holds that*

$$B(0, \frac{1}{4}) \subset f(\mathbb{B}).$$

The remaining part of this paper is organized as follows. In Sect. 2, we set up basic notations and give some preliminary results from the theory of octonionic slice regular functions. In Sect. 3, we first establish some useful lemmas and then use them to prove Theorem 1.1. Sect. 4 is devoted to the detailed proofs of Theorems 1.4, 1.5 and 1.6. In Sect. 5, we first use some ideas developed in the proof of Theorem 1.1 to prove Theorems 1.7 and 1.8. We then use Theorem 1.8 and a new convex combination identity (Proposition 3.5) to prove the growth and distortion theorems (Theorem 5.7) and Theorem 1.9. Finally, in Sect. 6 we use Julia lemma in [37] to prove a new and sharp boundary Schwarz lemma for quaternionic slice regular self-mappings of the open unit ball of the quaternions (Theorem 6.6) and give some consequences. This paper is closed with a comparison of these results and the corresponding results for holomorphic self-mappings of the open unit disc on the complex plane.

## 2. PRELIMINARIES

We recall in this section some necessary definitions and preliminary results on octonions and octonionic slice regular functions that we need later on.

**2.1. Octonions.** We denote by  $\mathbb{O}$  the non-commutative and non-associative division algebra of octonions (also called Cayley numbers). We refer to [29, 31, 41] for a more complete insight on octonions; here we shall just recall what is need for our purpose. A simple way to describe its construction is to consider a basis  $\mathcal{E} = \{e_0 = 1, e_1, \dots, e_6, e_7\}$  of  $\mathbb{R}^8$  and relations

$$(2.1) \quad e_i e_j = -\delta_{ij} + \psi_{ijk} e_k, \quad i, j, k = 1, 2, \dots, 7,$$

where  $\delta_{ij}$  is the Kronecker delta, and  $\psi_{ijk}$  is *totally antisymmetric* in  $i, j, k$ , non-zero and equal to one for the seven combinations in the following set

$$\Sigma = \{(1, 2, 3), (1, 4, 5), (2, 4, 6), (3, 4, 7), (5, 3, 6), (6, 1, 7), (7, 2, 5)\}$$

so that every element in  $\mathbb{O}$  can be uniquely written as  $w = x_0 + \sum_{k=1}^7 x_k e_k$ , with  $x_k$  ( $k = 1, 2, 3, 4$ ) being real numbers. The full multiplication table is conveniently encoded in a 7-point projective plane, the so-called Fano mnemonic graph, shown in Fig. 1 below. In the Fano mnemonic graph, the vertices are labeled by  $1, \dots, 7$  instead of  $e_1, \dots, e_7$ . Each of the 7 oriented lines gives a quaternionic triple. The product of any two imaginary units is given by the third unit on the unique line connecting them, with the sign determined by the relative orientation.

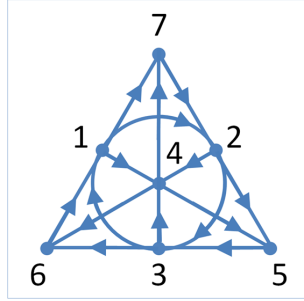


FIGURE 1. Fano Mnemonic

Alternatively,  $\mathbb{O}$  can be obtained from the quaternions  $\mathbb{H}$  by the well-known *Cayley-Dickson process*, which goes as follows. Let  $\{1, e_1, e_2, e_3 := e_1 e_2\}$  denote a real basis of  $\mathbb{H}$ . Each element  $w \in \mathbb{O}$  can be written as  $w = w_1 + w_2 e_4$ , where  $w_1, w_2 \in \mathbb{H}$  and  $e_4$  is a fixed imaginary unit of  $\mathbb{O}$ . The addition on  $\mathbb{O}$  is defined componentwisely and the product is defined by

$$(2.2) \quad zw = (z_1 + z_2 e_4)(w_1 + w_2 e_4) := z_1 w_1 - \bar{w}_2 z_2 + (z_2 \bar{w}_1 + w_2 z_1) e_4$$

for all  $z = z_1 + z_2 e_4$ ,  $w = w_1 + w_2 e_4 \in \mathbb{O}$ , where  $\bar{w}_1, \bar{w}_2$  are the conjugates of the quaternions  $w_1, w_2 \in \mathbb{H}$ . Set  $e_5 := e_1 e_4$ ,  $e_6 := e_2 e_4$ ,  $e_7 := e_3 e_4 = (e_1 e_2) e_4$ . Then  $\{1, e_1, e_2, \dots, e_7\}$  forms a real basis of  $\mathbb{O}$ , and one can easily verify that the product rule given by (2.2) is the same as the one in (2.1), and hence these two approaches indeed yield the same algebra  $\mathbb{O}$ .

For each  $w = x_0 + \sum_{k=1}^7 x_k e_k \in \mathbb{O}$ , the real number  $x_0$  is called the *real part* of  $w$ , and is denoted by  $\text{Re}(w)$ , while  $\sum_{k=1}^7 x_k e_k$  is called the *imaginary part* of  $w$  and is denoted by  $\text{Im}(w)$ . Moreover, we can define in a natural fashion the *conjugate*  $\bar{w} := x_0 - \sum_{k=1}^7 x_k e_k \in \mathbb{O}$ , and the *squared norm*  $|w|^2 := w\bar{w} = \bar{w}w = \sum_{k=0}^7 x_k^2$  (and by the Artin's theorem below,  $|zw| = |z||w|$  for any  $z, w \in \mathbb{O}$ ), which is induced by the standard Euclidean inner product on  $\mathbb{O} \cong \mathbb{R}^8$  given by

$$(2.3) \quad \langle z, w \rangle = \text{Re}(z\bar{w}) = \frac{1}{2}(z\bar{w} + w\bar{z}), \quad \forall z, w \in \mathbb{O}.$$

Also,

$$(2.4) \quad \langle z, w \rangle = \frac{1}{2}(|z+w|^2 - |z|^2 - |w|^2), \quad \forall z, w \in \mathbb{O}.$$

The *associator* of three octonions  $u, v, w \in \mathbb{O}$  is defined to be

$$[u, v, w] := (uv)w - u(vw),$$

which is *totally antisymmetric* in its arguments  $u, v, w \in \mathbb{O}$  and has *no real part*, i.e.

$$(2.5) \quad \operatorname{Re} [u, v, w] = 0.$$

Although the associator does not vanish in general, the octonions do satisfy a weak form of associativity known as *alternativity*, namely the so-called Moufang identities (cf. [27, p. 120]; also [31, p. 18]):

$$(2.6) \quad (uvu)w = u(v(uw)), \quad w(uvu) = ((wu)v)u, \quad u(vw)u = (uv)(wu).$$

The underlying reason for this is the so-called *Artin's theorem*, which can be stated as follows.

**Theorem 2.1.** (cf. [31, p. 18]) *In an alternative algebra  $A$ , every subalgebra generated by any two elements of  $A$  is always associative.*

For each  $\alpha \in \mathbb{O}$  with  $|\alpha| = 1$ , we now consider two multipliers  $\mathcal{L}_\alpha$  and  $\mathcal{R}_\alpha$  on the octonionic space  $(\mathbb{O}, \langle, \rangle)$  associated with  $\alpha$ , induced respectively by left and right multiplications, i.e.

$$\mathcal{L}_\alpha(w) = \alpha w, \quad \mathcal{R}_\alpha(w) = w\alpha, \quad \forall w \in \mathbb{O}.$$

Clearly,  $\mathcal{L}_\alpha$  and  $\mathcal{R}_\alpha$  are two  $\mathbb{R}$ -linear bijections with inverses  $\mathcal{L}_{\alpha^{-1}}$  and  $\mathcal{R}_{\alpha^{-1}}$ , respectively. Moreover, they are two unitary operators on  $(\mathbb{O}, \langle, \rangle)$  in virtue of equality (2.4). Therefore, we have the following simple lemma.

**Lemma 2.2.** *For each  $\alpha \in \mathbb{O}$  with  $|\alpha| = 1$ ,  $\mathcal{L}_\alpha$  and  $\mathcal{R}_\alpha$  are two unitary operators on the octonionic space  $(\mathbb{O}, \langle, \rangle)$ .*

As a direct consequence of the preceding lemma, we have the following result.

**Lemma 2.3.** *For any three octonions  $u, v, w \in \mathbb{O}$ , it holds that*

$$(2.7) \quad \langle u, [u, v, w] \rangle = 0.$$

*Proof.* First, we prove that

$$\langle I, [I, v, w] \rangle = 0$$

for every  $I \in \mathbb{S}$ ,  $v, w \in \mathbb{O}$ . Indeed,

$$\begin{aligned} \langle I, [I, v, w] \rangle &= \langle I, (Iv)w \rangle - \langle I, I(vw) \rangle \\ &= -\langle 1, ((Iv)w)I \rangle - \langle 1, vw \rangle && \text{by Lemma 2.2} \\ &= -\langle 1, [Iv, w, I] + (Iv)(wI) \rangle - \langle 1, vw \rangle \\ &= -\langle 1, (Iv)(wI) \rangle - \langle 1, vw \rangle && \text{by (2.5)} \\ &= -\langle 1, I(vw)I \rangle - \langle 1, vw \rangle && \text{by (2.6)} \\ &= \langle 1, vw \rangle - \langle 1, vw \rangle && \text{by Lemma 2.2} \\ &= 0. \end{aligned}$$

For each  $u \in \mathbb{O}$ . We write  $u = x + yI$  with  $x, y \in \mathbb{R}$  and  $I \in \mathbb{S}$ , then

$$\begin{aligned} \langle u, [u, v, w] \rangle &= \langle u, [yI, v, w] \rangle \\ &= y \langle x + yI, [yI, v, w] \rangle \\ &= y^2 \langle I, [I, v, w] \rangle \\ &= 0, \end{aligned}$$

which completes the proof. □



Despite the triviality of the above two lemmas, they turn out to be quite useful in our subsequent argument, especially in the proofs of Theorems 1.1, 1.4 and Proposition 5.1.

**2.2. Octonionic slice regular functions.** In order to introduce the notion of slice regularity on octonions  $\mathbb{O}$ , we rewrite each element  $w \in \mathbb{O}$  as  $w = x + yI$ , where  $x, y \in \mathbb{R}$  and

$$I = \frac{\operatorname{Im}(w)}{|\operatorname{Im}(w)|}$$

if  $\operatorname{Im}(w) \neq 0$ , otherwise we take  $I$  arbitrarily such that  $I^2 = -1$ . Then  $I$  is an element of the unit 6-dimensional sphere of purely imaginary octonions

$$\mathbb{S} = \{w \in \mathbb{O} : w^2 = -1\}.$$

For any two elements  $I, J \in \mathbb{S}$ , we define the *wedge product* of  $I$  and  $J$  as

$$I \wedge J := \frac{1}{2}[I, J] = \frac{1}{2}(IJ - JI),$$

which satisfies that

$$(2.8) \quad IJ = -\langle I, J \rangle + I \wedge J,$$

in view of (2.3). For every  $I \in \mathbb{S}$  we will denote by  $\mathbb{C}_I$  the plane  $\mathbb{R} \oplus I\mathbb{R}$ , isomorphic to  $\mathbb{C}$ , and, if  $\Omega \subseteq \mathbb{O}$ , by  $\Omega_I$  the intersection  $\Omega \cap \mathbb{C}_I$ . Also, we will denote by  $B(p, R)$  the Euclidean open ball of radius  $R$  centred at  $p \in \mathbb{O}$ , i.e.

$$B(p, R) = \{w \in \mathbb{O} : |w - p| < R\}.$$

For simplicity, we denote by  $\mathbb{B}$  the ball  $B(0, 1)$ .

We can now recall the notion of slice regularity.

**Definition 2.4.** Let  $\Omega$  be a domain in  $\mathbb{O}$ . A function  $f : \Omega \rightarrow \mathbb{O}$  is called (left) *slice regular* if, for all  $I \in \mathbb{S}$ , its restriction  $f_I$  to  $\Omega_I$  is *holomorphic*, i.e., it has continuous partial derivatives and satisfies

$$\bar{\partial}_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) = 0$$

for all  $x + yI \in \Omega_I$ .

A wide class of examples of slice regular functions is given by polynomials and power series of the variable  $w$  with octonionic coefficients on the right. Indeed, a function  $f$  is slice regular on an open ball  $B(0, R)$  if and only if  $f$  admits a power series expansion

$$(2.9) \quad f(w) = \sum_{n=0}^{\infty} w^n a_n,$$

which converges absolutely and uniformly on every compact subset of  $B(0, R)$  (see [17]). As shown in [5], the natural domains of definition of quaternionic slice regular functions are the so-called symmetric slice domains, which play for quaternionic slice regular functions the role played by domains of holomorphy for holomorphic functions of several complex variables. This is also the case for octonionic slice regular functions.

**Definition 2.5.** Let  $\Omega$  be a domain in  $\mathbb{O}$ .

1.  $\Omega$  is called a *slice domain* if it intersects the real axis and if for every  $I \in \mathbb{S}$ ,  $\Omega_I$  is a domain in  $\mathbb{C}_I$ .

2.  $\Omega$  is called an *axially symmetric domain* if for every point  $x + yI \in \Omega$ , with  $x, y \in \mathbb{R}$  and  $I \in \mathbb{S}$ , the entire 6-dimensional sphere  $x + y\mathbb{S}$  is contained in  $\Omega$ .

A domain in  $\mathbb{O}$  is called a *symmetric slice domain* if it is not only a slice domain, but also an axially symmetric domain. By the very definition, an open ball  $B(0, R)$  is a typical symmetric slice domain. For slice regular functions a natural definition of slice derivative is given as follows:

**Definition 2.6.** Let  $f : \Omega \rightarrow \mathbb{O}$  be a slice regular function. For each  $I \in \mathbb{S}$ , the  $I$ -derivative of  $f$  at  $w = x + yI$  is defined by

$$\partial_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) f_I(x + yI)$$

on  $\Omega_I$ . The *slice derivative* of  $f$  is the function  $f'$  defined by  $\partial_I f$  on  $\Omega_I$  for all  $I \in \mathbb{S}$ .

From the very definition and Artin's theorem for alternative algebras (Theorem 2.1) mentioned as before, the slice derivative of a slice regular function  $f : \Omega \rightarrow \mathbb{O}$  is still slice regular so that we can iterate the differentiation to obtain the  $n$ -th slice derivative

$$\partial_I^n f(w) = \frac{\partial^n f}{\partial x^n}(w), \quad \forall n \in \mathbb{N},$$

where  $w = x + yI \in \Omega$ . In what follows, for the sake of simplicity, we will denote the  $n$ -th slice derivative by  $f^{(n)}$  for every  $n \in \mathbb{N}$ . Now it is easy to see that the coefficient  $a_n$  appeared in (2.9) is exactly  $f^{(n)}(0)/n!$ . We will also omit the term 'slice' when referring to slice regular functions.

Since slice regularity does not keep under pointwise product of two regular functions, a new multiplication operation, called the regular product (or  $*$ -product), will be defined by means of a suitable modification of the usual one subject to the non-commutativity and the non-associativity of  $\mathbb{O}$ , based on the following splitting lemma (compare [17, Lemma 2.7]; also [22, Lemma 2.4]), which is more convenient for our subsequent arguments.

**Lemma 2.7.** Let  $f$  be a regular function on a domain  $\Omega \subseteq \mathbb{O}$ . Then for any  $I, J, K \in \mathbb{S}$  with  $I, J, IJ, K$  being mutually perpendicular with respect to the standard Euclidean inner product on  $\mathbb{O}$ , there exist four holomorphic functions  $F_k : \Omega_I \rightarrow \mathbb{C}_I$ ,  $k = 1, 2, 3, 4$  such that

$$(2.10) \quad f_I(z) = F_1(z) + F_2(z)J + (F_3(z) + \overline{F_4(z)}J)K$$

for all  $z \in \Omega_I$ .

*Proof.* The well-known Cayley-Dickson process guarantees the existence of four functions  $F_k : \Omega_I \rightarrow \mathbb{C}_I$ ,  $k = 1, 2, 3, 4$  such that equality (2.10) holds for all  $z \in \Omega_I$ . Now it remains to verify the holomorphy of these four functions  $F_k$ ,  $k = 1, 2, 3, 4$ . By (2.2), for every  $z \in \Omega_I$ ,

$$(2.11) \quad \begin{aligned} \bar{\partial}_I f(z) &= \bar{\partial}_I F_1(z) + \bar{\partial}_I F_2(z)J + \left( (F_3(z) + \overline{F_4(z)}J) \bar{\partial}_I \right) K \\ &= \bar{\partial}_I F_1(z) + \bar{\partial}_I F_2(z)J + (\bar{\partial}_I F_3(z) + \partial_I \overline{F_4(z)}J)K. \end{aligned}$$

Thus  $\bar{\partial}_I f(z) = 0$  implies  $\bar{\partial}_I F_k(z) = 0$ , proving the holomorphy of these four functions  $F_k$ ,  $k = 1, 2, 3, 4$ . The proof is complete.  $\square$

Let  $\Omega \subseteq \mathbb{O}$  be a symmetric slice domain and  $I, J, K \in \mathbb{S}$  be such that  $I, J, IJ, K$  are mutually perpendicular with respect to the standard Euclidean inner product on  $\mathbb{O}$ . Let  $f$  and  $g$  be two regular functions on  $\Omega \subseteq \mathbb{O}$ . Then the splitting lemma above guarantees the existence of eight holomorphic functions  $F_k, G_k : \Omega_I \rightarrow \mathbb{C}_I$ ,  $k = 1, 2, 3, 4$  such that for all  $z \in \Omega_I$ ,

$$f_I(z) = F_1(z) + F_2(z)J + (F_3(z) + \overline{F_4(z)}J)K$$

and

$$g_I(z) = G_1(z) + G_2(z)J + (G_3(z) + \overline{G_4(z)}J)K$$

Following the approach in [5], we define the function  $f_I * g_I : \Omega_I \rightarrow \mathbb{O}$  as

$$(2.12) \quad f_I * g_I(z) := H_1(z) + H_2(z)J + (H_3(z) + \overline{H_4(z)}J)K,$$

where

$$\begin{aligned} H_1(z) &= F_1(z)G_1(z) - F_2(z)\overline{G_2(z)} - F_3(z)\overline{G_3(z)} - F_4(z)\overline{G_4(z)}, \\ H_2(z) &= F_1(z)G_2(z) + F_2(z)\overline{G_1(z)} + \overline{F_3(z)}G_4(z) - \overline{F_4(z)}\overline{G_3(z)}, \\ H_3(z) &= F_1(z)G_3(z) - \overline{F_2(z)}\overline{G_4(z)} + \overline{F_3(z)}G_1(z) + \overline{F_4(z)}\overline{G_2(z)}, \\ H_4(z) &= F_1(z)G_4(z) + \overline{F_2(z)}\overline{G_3(z)} - \overline{F_3(z)}\overline{G_2(z)} + F_4(z)\overline{G_1(z)}. \end{aligned}$$

Then  $f_I * g_I(z)$  is obviously holomorphic satisfying  $f_I * g_I(z) = f(z)g(z)$  (independent of the choice of  $I \in \mathbb{S}$ ) for all  $z \in \Omega \cap \mathbb{R}$ . Therefore,  $f_I * g_I$  admits a *unique* regular extension to  $\Omega$ , independent of the choice of  $I \in \mathbb{S}$ , via the formula (2) in [19, Proposition 6], analogous to [5, Lemma 4.4]. We denote by  $\text{ext}(f_I * g_I)$  this unique regular extension of  $f_I * g_I$ .

**Definition 2.8.** Let  $f$  and  $g$  be two regular functions on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$ . Then the regular function

$$f * g(w) := \text{ext}(f_I * g_I)(w)$$

defined as the extension of (2.12) is called the *regular product* (or *\*-product*) of  $f$  and  $g$ .

*Remark 2.9.* In the special case that  $\Omega = B(0, R)$ , there is a more direct way of defining the regular product. Let  $f, g : B(0, R) \rightarrow \mathbb{O}$  be two regular functions and let

$$f(w) = \sum_{n=0}^{\infty} w^n a_n, \quad g(w) = \sum_{n=0}^{\infty} w^n b_n$$

be their power series expansions. The regular product of  $f$  and  $g$  given in Definition 2.8 is coherent with the one given by

$$f * g(w) := \sum_{n=0}^{\infty} w^n \left( \sum_{k=0}^n a_k b_{n-k} \right).$$

This follows from the identity principle and the fact that these two products defined in these two ways are exactly the usual pointwise product when they are restricted to  $B(0, R) \cap \mathbb{R}$ , as one can patiently verify.

*Remark 2.10.* When  $\Omega \subseteq \mathbb{O}$  is a symmetric slice domain, the same reason as in the preceding remark also shows that the product defined in [19, Definition 9] coincides with the one in Definition 2.8 provided all the considered functions are regular.

*Remark 2.11.* Notice that the regular product is obviously distributive, but in general non-commutative and non-associative, since the underlying algebra  $\mathbb{O}$  is non-commutative and non-associative. However, it is commutative and associative in some special cases. For instance, let  $f$  and  $g$  be two regular functions on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$  and satisfy the additional condition that  $f(\Omega_I) \subseteq \mathbb{C}_I$  and  $g(\Omega_I) \subseteq \mathbb{C}_I$  for some  $I \in \mathbb{S}$ , then from Artin's theorem for alternative algebras (Theorem 2.1), Remark 2.9 and the identity principle it follows that

$$f * g = g * f$$

and

$$(f * g) * h = f * (g * h)$$

for every regular function  $h$  on  $\Omega$ . Moreover, if  $f$  is further slice preserving (i.e.  $f(\Omega_I) \subseteq \mathbb{C}_I$  for every  $I \in \mathbb{S}$ ), then

$$f * h = fh = h * f.$$

Again let  $\Omega \subseteq \mathbb{O}$  be a symmetric slice domain and  $I, J, K \in \mathbb{S}$  be such that  $I, J, IJ, K$  are mutually perpendicular with respect to the standard Euclidean inner product on  $\mathbb{O}$ . Let  $f$  be a regular function on  $\Omega \subseteq \mathbb{O}$ . Then the splitting lemma (Lemma 2.7) guarantees the existence of four holomorphic functions  $F_k : \Omega_I \rightarrow \mathbb{C}_I$ ,  $k = 1, 2, 3, 4$  such that for all  $z \in \Omega_I$ ,

$$f_I(z) = F_1(z) + F_2(z)J + (F_3(z) + \overline{F_4(z)}J)K.$$

We define two functions  $f_I^c, f_I^s : \Omega_I \rightarrow \mathbb{O}$  as

$$(2.13) \quad f_I^c(z) := \overline{F_1(\bar{z})} - F_2(z)J - (F_3(z) + \overline{F_4(z)}J)K,$$

and

$$(2.14) \quad f_I^s(z) := f_I * f_I^c(z) = \sum_{k=1}^4 F_k(z) \overline{F_k(\bar{z})} = f_I^c * f_I(z).$$

Here the second equality in (2.14) follows from (2.12). Then both  $f_I^c(z)$  and  $f_I^s(z)$  are obviously holomorphic satisfying  $f_I^c(z) = \overline{f_I(z)}$  and  $f_I^s(z) = |f(z)|^2$  (independent of the choice of  $I \in \mathbb{S}$ ) for all  $z \in \Omega \cap \mathbb{R}$ . Therefore, they admit respectively a *unique* regular extension to  $\Omega$ , independent of the choice of  $I \in \mathbb{S}$ . We denote them by  $\text{ext}(f_I^c)$  and  $\text{ext}(f_I^s)$ , respectively.

**Definition 2.12.** Let  $f$  be a regular function on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$ . Then the regular function

$$f^c(w) := \text{ext}(f_I^c)(w)$$

defined as the extension of (2.13) is called the *regular conjugate* of  $f$ , and the regular function

$$f^s(w) := \text{ext}(f_I^s)(w) = f * f^c(w) = f^c * f(w)$$

is called the *symmetrization* of  $f$ .

*Remark 2.13.* As before, one can also prove that for octonionic regular functions on symmetric slice domains in  $\Omega \subseteq \mathbb{O}$ , the notions given in Definition 2.12 for regular conjugate and symmetrization coincide essentially with those introduced in [19, Definition 11]. In the special case that  $\Omega = B(0, R)$ , there is also an equivalent way of defining the regular conjugate and the symmetrization of regular functions, which goes as follows. Let  $f : B(0, R) \rightarrow \mathbb{O}$  be a regular function with the power series expansion

$$f(w) = \sum_{n=0}^{\infty} w^n a_n.$$

Then the regular conjugate and the symmetrization of  $f$  are respectively given by

$$f^c(w) = \sum_{n=0}^{\infty} w^n \bar{a}_n,$$

and

$$f^s(w) = f * f^c(w) = f^c * f(w) = \sum_{n=0}^{\infty} w^n \left( \sum_{k=0}^n a_k \bar{a}_{n-k} \right).$$

One can easily verify that these two definitions are the same as those in Definition 2.12.

*Remark 2.14.* From (2.14) one immediately deduces that the symmetrization  $f^s$  of every regular function  $f$  on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$  is slice preserving, i.e.  $f^s(\Omega_I) \subseteq \mathbb{C}_I$  for every  $I \in \mathbb{S}$ .

Both the regular conjugate and the symmetrization are well-behaved with respect to the regular product.

**Proposition 2.15.** *Let  $f$  and  $g$  be two regular functions on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$ . Then  $(f^c)^c = f$ ,  $(f * g)^c = g^c * f^c$  and*

$$(2.15) \quad (f * g)^s = f^s g^s = g^s f^s.$$

*Proof.* We only prove (2.15), since the remaining is obvious in virtue of (2.12) and (2.13). The power series case of (2.15) was proved in [20, Lemma 2], and the general case follows immediately from the former case and the identity principle.  $\square$

Now we can use the notions of regular conjugate and symmetrization introduced above to define the regular reciprocal of a regular function:

**Definition 2.16.** Let  $f$  be a non-identically vanishing regular function on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$  and  $\mathcal{Z}_{f^s}$  the set of zeros of its symmetrization  $f^s$ . We define the *regular reciprocal* of  $f$  as the regular function  $f^{-*} : \Omega \setminus \mathcal{Z}_{f^s} \rightarrow \mathbb{O}$  given by

$$(2.16) \quad f^{-*}(w) := f^s(w)^{-1} f^c(w).$$

The function  $f^{-*}$  defined in (2.16) deserves the name of regular reciprocal of  $f$  due to the following:

**Proposition 2.17.** *Let  $f$  be a non-identically vanishing regular function on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$  and  $\mathcal{Z}_{f^s}$  the set of zeros of its symmetrization  $f^s$ . Then*

$$f^{-*} * f = f * f^{-*} = 1$$

on  $\Omega \setminus \mathcal{Z}_{f^s}$ .

We conclude this section with the following simple proposition.

**Proposition 2.18.** *Let  $f$  and  $g$  be two non-identically vanishing regular function on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$ . Then*

$$(f * g)^{-*} = g^{-*} * f^{-*}$$

on  $\Omega \setminus (\mathcal{Z}_{f^s} \cup \mathcal{Z}_{g^s})$ .

*Proof.* The result follows from Remarks 2.11 and 2.14, together with Proposition 2.15:

$$(f * g)^{-*} = (f^s g^s)^{-1} (g^c * f^c) = ((g^s)^{-1} g^c) * ((f^s)^{-1} f^c) = g^{-*} * f^{-*}.$$

$\square$

### 3. PROOF OF THEOREM 1.1

In this section, we shall give a proof of Theorem 1.1. Before presenting the details, we need some auxiliary results.

Let  $\Omega \subseteq \mathbb{O}$  be a symmetric slice domain. For each regular function  $f : \Omega \rightarrow \mathbb{O}$  and each  $\xi \in \Omega$ , an argument similar to the one in the proof of [14, Proposition 3.17], together with the splitting lemma (Lemma 2.7) and Artin's theorem for alternative algebras (Theorem 2.1), guarantees the existence of a unique regular function on  $\Omega$ , denoted by  $R_\xi f$ , such that

$$(3.1) \quad f(w) - f(\xi) = (w - \xi) * R_\xi f(w), \quad \forall w \in \Omega.$$

Applying the same procedure to  $R_\xi f$  at the point  $\bar{\xi}$  yields

$$(3.2) \quad \begin{aligned} f(w) &= f(\xi) + (w - \xi) * \left( R_\xi f(\bar{\xi}) + (w - \bar{\xi}) * R_{\bar{\xi}} R_\xi f(w) \right) \\ &= f(\xi) + (w - \xi) R_\xi f(\bar{\xi}) + \Delta_\xi(w) R_{\bar{\xi}} R_\xi f(w), \quad \forall w \in \Omega, \end{aligned}$$

where  $\Delta_\xi(w) := (w - \xi) * (w - \bar{\xi}) = w^2 - 2w\operatorname{Re}(\xi) + |\xi|^2$  is called the *characteristic polynomial* of  $\xi$  or the *symmetrization* of  $w - \xi$ , and the second equality in (3.2) follows from Remark 2.11.

From the very definition and (3.1), one can see that  $R_\xi f(\bar{\xi})$  is exactly the *sphere derivative*  $\partial_s f(\xi)$  of  $f$  at the point  $\xi$ :

$$\partial_s f(\xi) := (2\operatorname{Im}(\xi))^{-1} (f(\xi) - f(\bar{\xi})) = R_\xi f(\bar{\xi}).$$

In addition, for every  $v \in \partial\mathbb{B}$  and every  $t \in \mathbb{R}$  small enough, replacing  $w$  by  $\xi + tv$  in (3.2) yields

$$f(\xi + tv) - f(\xi) = tv\partial_s f(\xi) + t(tv^2 + (\xi v - v\bar{\xi})) R_{\bar{\xi}} R_\xi f(\xi + tv),$$

from which the following lemma immediately follows.

**Lemma 3.1.** *Let  $f$  be a regular function on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$  and  $\xi \in \Omega$ . Then for every  $v \in \partial\mathbb{B}$ , the directional derivative of  $f$  along  $v$  at  $\xi$  is given by*

$$(3.3) \quad \frac{\partial f}{\partial v}(\xi) := \lim_{\mathbb{R} \ni t \rightarrow 0} \frac{f(\xi + tv) - f(\xi)}{t} = v\partial_s f(\xi) + (\xi v - v\bar{\xi}) R_{\bar{\xi}} R_\xi f(\xi).$$

In particular, it holds that

$$(3.4) \quad f'(\xi) = \partial_s f(\xi) + 2\operatorname{Im}(\xi) R_{\bar{\xi}} R_\xi f(\xi).$$

Also, the following lemma is needed in the proof of Theorem 1.1.

**Lemma 3.2.** *Let  $f$  be a regular self-mapping of the open unit ball  $\mathbb{B}$ . Then the inequality*

$$(3.5) \quad \frac{1 - |f(w)|^2}{1 - |w|^2} \geq \frac{|1 - \langle f(0), f(w) \rangle|^2}{1 - |f(0)|^2}$$

holds for all  $w \in \mathbb{B}$ .

*Proof.* Fix an arbitrary point  $w \in \mathbb{B}$ , let  $I \in \mathbb{S}$  be such that  $w \in \mathbb{B}_I$ . Then by the splitting lemma (Lemma 2.7), we can find  $J$  and  $K$  in  $\mathbb{S}$ , such that  $I, J, IJ, K$  are mutually perpendicular with respect to the standard Euclidean inner product on  $\mathbb{O}$  and there are four holomorphic functions  $F_k : \mathbb{B}_I \rightarrow \mathbb{B}_I$ ,  $k = 1, 2, 3, 4$ , such that

$$(3.6) \quad f_I(z) = F_1(z) + F_2(z)J + (F_3(z) + \overline{F_4(z)}J)K, \quad \forall z \in \mathbb{B}_I.$$

Let  $B^4 \subset \mathbb{C}^4$  be the open unit ball. We consider the holomorphic mapping  $F : \mathbb{B}_I \rightarrow \mathbb{C}_I^4$  given by

$$F(z) := (F_1(z), F_2(z), F_3(z), F_4(z)),$$

which maps  $\mathbb{B}_I$  into  $B^4$  in virtue of the fact that

$$|F(z)|^2 = \sum_{k=1}^4 |F_k(z)|^2 = |f(z)|^2 < 1$$

for all  $z \in \mathbb{B}_I$ . Now it follows from the classical Schwarz-Pick lemma (see e.g. [38, Theorem 8.1.4]) that

$$(3.7) \quad \frac{|1 - \langle F(0), F(z) \rangle_{\mathbb{C}_I^4}|^2}{(1 - |F(0)|^2)(1 - |F(z)|^2)} \leq \frac{1}{1 - |z|^2}, \quad \forall z \in \mathbb{B}_I,$$

where  $\langle \cdot, \cdot \rangle_{\mathbb{C}_I^4}$  denotes the standard Hermitian inner product on  $\mathbb{C}_I^4$ , i.e. for any two vectors  $\alpha = (\alpha_1, \dots, \alpha_4)$ ,  $\beta = (\beta_1, \dots, \beta_4) \in \mathbb{C}_I^4$ ,

$$\langle \alpha, \beta \rangle_{\mathbb{C}_I^4} = \sum_{k=1}^4 \alpha_k \bar{\beta}_k.$$

Once notice that  $\langle f(0), f(w) \rangle = \operatorname{Re}(\langle F(0), F(z) \rangle_{\mathbb{C}_I^4}) \in \mathbb{R}$ , inequality (3.5) immediately follows from (3.7).  $\square$

Now we come to prove Theorem 1.1.

*Proof of Theorem 1.1.* We first prove the assertion (i). The proof of the first equality in (1.2) is essentially the same as the corresponding part in the proof of [37, Theorem 4]. First, it follows from inequality (3.5) that the directional derivative of  $|f|^2$  along  $\xi$  at the boundary point  $\xi \in \partial\mathbb{B}$  satisfies that

$$(3.8) \quad \frac{\partial |f|^2}{\partial \xi}(\xi) \geq 2 \frac{|1 - \langle f(0), f(\xi) \rangle|^2}{1 - |f(0)|^2}.$$

However,

$$(3.9) \quad \frac{\partial |f|^2}{\partial \tau}(\xi) = 0, \quad \forall \tau \in T_\xi(\partial\mathbb{B}) \cong \mathbb{R}^7.$$

Indeed, for each unit tangent vector  $\tau \in T_\xi(\partial\mathbb{B})$ , take a smooth curve  $\gamma : (-1, 1) \rightarrow \bar{\mathbb{B}}$  such that

$$\gamma(0) = \xi, \quad \gamma'(0) = \tau.$$

By definition we have

$$\frac{\partial |f|^2}{\partial \tau}(\xi) = \left( \frac{d}{dt} |f(\gamma(t))|^2 \right) \Big|_{t=0} = 0,$$

since the function  $|f(\gamma(t))|^2$  in  $t$  attains its maximum at the point  $t = 0$ .

In view of Lemma 3.1, we have

$$\frac{\partial f}{\partial v}(\xi) = v \partial_s f(\xi) + (\xi v - v \bar{\xi}) R_{\bar{\xi}} R_\xi f(\xi), \quad \forall v \in \partial\mathbb{B},$$

from which and Lemma 2.2 it follows that

$$(3.10) \quad \begin{aligned} \frac{\partial |f|^2}{\partial v}(\xi) &= 2 \left\langle \frac{\partial f}{\partial v}(\xi), f(\xi) \right\rangle \\ &= 2 \left\langle v \partial_s f(\xi) + (\xi v - v \bar{\xi}) R_{\bar{\xi}} R_\xi f(\xi), f(\xi) \right\rangle \\ &= 2 \left\langle v, f(\xi) \overline{\partial_s f(\xi)} + \bar{\xi} \left( f(\xi) \overline{R_{\bar{\xi}} R_\xi f(\xi)} \right) - \left( f(\xi) \overline{R_{\bar{\xi}} R_\xi f(\xi)} \right) \xi \right\rangle \\ &=: 2(A + B), \end{aligned}$$

where

$$(3.11) \quad \begin{aligned} A &= \left\langle v, f(\xi) \left( \overline{\partial_s f(\xi)} - \bar{\xi} \overline{R_{\bar{\xi}} R_\xi f(\xi)} \right) \right\rangle \\ &= \left\langle v, f(\xi) \left( \overline{f'(\xi)} - \bar{\xi} \overline{R_{\bar{\xi}} R_\xi f(\xi)} \right) \right\rangle, \end{aligned}$$

and

$$(3.12) \quad B = \left\langle v, \bar{\xi} \left( f(\xi) \overline{R_{\bar{\xi}} R_\xi f(\xi)} \right) - [f(\xi), \overline{R_{\bar{\xi}} R_\xi f(\xi)}, \xi] \right\rangle.$$

The second equality in (3.11) follows from equality (3.4). Substituting the following simple equalities

$$\begin{aligned} f(\xi) \left( \overline{R_{\bar{\xi}} R_{\xi} f(\xi)} \bar{\xi} \right) &= \left( f(\xi) \overline{R_{\bar{\xi}} R_{\xi} f(\xi)} \right) \bar{\xi} - [f(\xi), \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}, \bar{\xi}] \\ &= \left( f(\xi) \overline{R_{\bar{\xi}} R_{\xi} f(\xi)} \right) \bar{\xi} + [f(\xi), \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}, \xi] \end{aligned}$$

into the second equality in (3.11) yields

$$(3.13) \quad A = \left\langle v, f(\xi) \overline{f'(\xi)} - \left( f(\xi) \overline{R_{\bar{\xi}} R_{\xi} f(\xi)} \right) \bar{\xi} - [f(\xi), \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}, \xi] \right\rangle.$$

Substituting (3.12) and (3.13) into (3.10) gives

$$(3.14) \quad \begin{aligned} \frac{\partial |f|^2}{\partial v}(\xi) &= 2 \left\langle v, f(\xi) \overline{f'(\xi)} + [\bar{\xi}, f(\xi) \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}] - 2[f(\xi), \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}, \xi] \right\rangle \\ &= 2 \left\langle v, f(\xi) \overline{f'(\xi)} + [\bar{\xi}, f(\xi) \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}] + 2[\xi, f(\xi), R_{\bar{\xi}} R_{\xi} f(\xi)] \right\rangle. \end{aligned}$$

Now it follows from (3.9) and (3.14) that

$$f(\xi) \overline{f'(\xi)} + [\bar{\xi}, f(\xi) \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}] + 2[\xi, f(\xi), R_{\bar{\xi}} R_{\xi} f(\xi)] \perp T_{\xi}(\partial \mathbb{B})$$

so that in view of (3.8) and (3.14),

$$\begin{aligned} \frac{\partial |f|}{\partial \xi}(\xi) &= \bar{\xi} \left( f(\xi) \overline{f'(\xi)} + [\bar{\xi}, f(\xi) \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}] + 2[\xi, f(\xi), R_{\bar{\xi}} R_{\xi} f(\xi)] \right) \\ &\geq \frac{|1 - \langle f(0), f(\xi) \rangle|^2}{1 - |f(0)|^2}, \end{aligned}$$

which completes the proof of (1.2).

To prove (1.3), notice first that the first equality in (1.3) directly follows from (1.2). It remains to prove the following inequality

$$\frac{\partial |f|}{\partial \xi}(\xi) \geq \frac{2}{1 + \operatorname{Re} f'(0)}.$$

To this end, let  $I \in \mathbb{S}$  be such that  $\xi \in \partial \mathbb{B} \cap \mathbb{C}_I$ . Then by the splitting lemma (Lemma 2.7), we can find  $J$  and  $K$  in  $\mathbb{S}$ , such that  $I, J, IJ, K$  are mutually perpendicular and if  $\mathcal{H}$  is the subspace of  $\mathbb{O}$  generated by  $\{1, I, J, IJ\}$ , then there are two regular functions  $F : \mathbb{B} \cap \mathcal{H} \rightarrow \mathbb{B} \cap \mathcal{H}$  and  $G : \mathbb{B} \cap \mathcal{H} \rightarrow \mathbb{B} \cap \mathcal{H}K$  such that

$$f(w) = F(w) + G(w), \quad \forall w \in \mathbb{B} \cap \mathcal{H}.$$

Then for each  $w \in \mathbb{B} \cap \mathcal{H}$ , we have

$$(3.15) \quad |f(w)|^2 = |F(w)|^2 + |G(w)|^2$$

and

$$f'(w) = F'(w) + G'(w).$$

Moreover,

$$F(\xi) = \xi, \quad G(\xi) = 0, \quad \operatorname{Re} f'(0) = \operatorname{Re} F'(0).$$

Now it follows from Corollary 6.10 below that

$$\frac{\partial |f|}{\partial \xi}(\xi) = \frac{\partial |F|}{\partial \xi}(\xi) = F'(\xi) - [\xi, R_{\bar{\xi}} R_{\xi} F(\xi)] \geq \frac{2}{1 + \operatorname{Re} F'(0)} = \frac{2}{1 + \operatorname{Re} f'(0)}.$$

If equality holds for inequality in (1.3), then it again follows from Corollary 6.10 below that

$$(3.16) \quad F(w) = w(1 - wa\bar{\xi})^{-*} * (w - a\xi)\bar{\xi} \quad \forall w \in \mathbb{B} \cap \mathcal{H},$$



for some constant  $a \in [-1, 1)$ . Furthermore, it follows from equality in (3.15) that

$$|G(w)|^2 = |f(w)|^2 - |F(w)|^2 \leq 1 - |F(w)|^2, \quad \forall w \in \mathbb{B} \cap \mathcal{H},$$

which together with (3.16) implies that  $G \equiv 0$ , in virtue of the maximum principle (Theorem 4.3 below), and hence

$$f(w) = \text{ext } F(w) = w(1 - wa\bar{\xi})^{-*} * (w\bar{\xi} - a), \quad \forall w \in \mathbb{B}.$$

Therefore, the equality in inequality (1.3) can hold only for regular self-mappings of the form (1.4), and a direct calculation shows that it does indeed hold for all such regular self-mappings. Now the proof is complete.  $\square$

*Remark 3.3.* It is worth remarking here that, as in the quaternionic setting, the Lie bracket in (1.3) does not necessarily vanish and  $f'(\xi)$  may not be a real number. The following example comes an explicit counterexample.

*Example 3.4.* Fix two imaginary units  $I, J \in \mathbb{S}$  with  $I \perp J$ . Set

$$\varphi(w) = w(1 + wI/2)^{-*} * (I/2 - w).$$

Then the restriction  $\varphi_I$  of  $\varphi$  to  $\mathbb{B}_I$  is a holomorphic Blaschke product of order 2 so that  $\varphi$  is a regular self-mapping of  $\mathbb{B}$ , in virtue of Proposition 3.5. Define another regular function  $f$  on  $\mathbb{B}$  given by

$$f(w) = \varphi(w) * J = w(w^2 + 4)^{-1} (2(w^2 + 1)(IJ) - 3wJ).$$

We claim that  $f$  maps  $\mathbb{B}$  into  $\mathbb{B}$ . We argue by contradiction and suppose that there is a point  $\omega_0 \in \mathbb{O} \setminus \mathbb{B}$  such that  $f - \omega_0 = (\varphi + \omega_0 J) * J$  has a zero in  $\mathbb{B}$ . By [19, Corollary 25],

$$\bigcup_{w \in \mathcal{Z}_{f-\omega_0}} \mathbb{S}_w = \bigcup_{w \in \mathcal{Z}_{\varphi+\omega_0 J}} \mathbb{S}_w.$$

This shows that  $\varphi + \omega_0 J$  also has a zero in  $\mathbb{B}$ , contradicting the fact that  $\varphi(\mathbb{B}) \subseteq \mathbb{B}$ . Therefore,  $f(\mathbb{B}) \subseteq \mathbb{B}$ . Moreover, it is evident that  $f$  is regular on  $\overline{\mathbb{B}}$  satisfying both  $f(0) = 0$  and  $f(J) = J$ . Thus  $f$  verifies all the assumptions in Theorem 1.1 (ii). However, we find that  $f'(J)$  is indeed not a real number. In fact, a straightforward calculation shows that

$$f'(J) = \frac{4}{3}(2 - IJ) \notin \mathbb{R}, \quad R_{-J}R_J f(J) = \frac{2}{3}(I - 2J),$$

while

$$f'(J) - [J, R_{-J}R_J f(J)] = \frac{8}{3} > 1$$

as predicated by Theorem 1.1 (ii). One can also shows that

$$\frac{\partial |f|}{\partial J}(J) = \frac{8}{3}$$

using the obvious fact that  $f_J(w) = \varphi_J(w)J$  for all  $w \in \mathbb{B}_J$ , together with Proposition 3.5.

The regular functions of the form (1.4) are indeed self-mappings of the open unit ball  $\mathbb{B} \subset \mathbb{O}$ , due to the following result:

**Proposition 3.5.** *Let  $f$  be a regular function on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$  such that  $f(\Omega_I) \subseteq \mathbb{C}_I$  for some  $I \in \mathbb{S}$ . Then the convex combination identity*

$$(3.17) \quad |f(x + yJ)|^2 = \frac{1 + \langle I, J \rangle}{2} |f(x + yI)|^2 + \frac{1 - \langle I, J \rangle}{2} |f(x - yI)|^2$$

holds for every  $x + yJ \in \Omega$ .

*Proof.* The idea is essentially the same as in [36]. Let  $I \in \mathbb{S}$  be as described in the proposition. First, it is easy to verify that for every  $J \in \mathbb{S}$ , the set  $\{1, I, I \wedge J, I(I \wedge J)\}$  is an orthogonal set of  $\mathbb{O} \simeq \mathbb{R}^8$ . By the representation formula for regular functions (cf. [19, Proposition 6]),

$$(3.18) \quad f(w) = \frac{1}{2}(f(z) + f(\bar{z})) - \frac{1}{2}J(I(f(z) - f(\bar{z})))$$

for every  $w = x + yJ \in \Omega$  with  $z = x + yI$  and  $\bar{z} = x - yI$ . By assumption,  $f(\Omega_I) \subseteq \mathbb{C}_I$ . This together with Artin's theorem for alternative algebras (Theorem 2.1) allows us to rewrite equality (3.18) as

$$(3.19) \quad f(w) = \frac{1}{2}(f(z) + f(\bar{z})) - \frac{1}{2}(JI)(f(z) - f(\bar{z}))$$

Taking modulus on both sides of (3.19) and applying Lemma 2.2 to obtain

$$(3.20) \quad \begin{aligned} |f(w)|^2 &= \frac{1}{4} \left( |f(z) + f(\bar{z})|^2 + |f(z) - f(\bar{z})|^2 \right) \\ &\quad - \frac{1}{2} \left\langle f(z) + f(\bar{z}), (JI)(f(z) - f(\bar{z})) \right\rangle \\ &= \frac{1}{2} \left( |f(z)|^2 + |f(\bar{z})|^2 \right) - \frac{1}{2} \left\langle (f(z) + f(\bar{z}))(\overline{f(z)} - \overline{f(\bar{z})}), JI \right\rangle \\ &=: \frac{1}{2} \left( |f(z)|^2 + |f(\bar{z})|^2 \right) - \frac{1}{2} A, \end{aligned}$$

where

$$(3.21) \quad A = \left\langle (f(z) + f(\bar{z}))(\overline{f(z)} - \overline{f(\bar{z})}), JI \right\rangle.$$

Recalling equality in (2.8), an orthogonality consideration gives

$$(3.22) \quad \begin{aligned} A &= -\langle I, J \rangle \left\langle (f(z) + f(\bar{z}))(\overline{f(z)} - \overline{f(\bar{z})}), 1 \right\rangle \\ &= -\langle I, J \rangle \left\langle f(z) + f(\bar{z}), f(z) - f(\bar{z}) \right\rangle \\ &= -\langle I, J \rangle \left( |f(z)|^2 - |f(\bar{z})|^2 \right). \end{aligned}$$

Now the desired equality (3.17) immediately follows by substituting (3.22) into (3.20). The proof is complete.  $\square$

*Remark 3.6.* Together with Proposition 3.5, the argument used in Example 3.4 also shows that the regular functions  $f$  of the form

$$f(w) = (1 - w\bar{u})^{-*} * (w - u) * v$$

with  $u \in \mathbb{B}$  and  $v \in \partial\mathbb{B}$  are regular self-mappings of  $\mathbb{B}$ .

## 4. PROOFS OF THEOREMS 1.4, 1.5 AND 1.6

We begin with a notion of regular diameter, which is intimately related to a new regular composition (cf. [35]).

**Definition 4.1.** Let  $u \in \mathbb{O}$  and  $f : \mathbb{B} \rightarrow \mathbb{O}$  a regular function with Taylor expansion

$$f(w) = \sum_{n=0}^{\infty} w^n a_n.$$

We define the *regular composition* of  $f$  with the regular function  $w \mapsto wu$  to be

$$f_u(w) := \sum_{n=0}^{\infty} (wu)^{*n} * a_n = \sum_{n=0}^{\infty} w^n (u^n a_n).$$

If  $|u| = 1$ , the radius of convergence of the series expansion for  $f_u$  is the same as that for  $f$ . Moreover, if  $u$  and  $w_0$  lie in the same plane  $\mathbb{C}_I$ , then  $u$  and  $w_0$  commute, and hence  $f_u(w_0) = f(uw_0)$ . In particular, if  $u \in \mathbb{R}$ , then  $f_u(w) = f(uw)$  for every  $w \in \mathbb{B}$ .

**Definition 4.2.** Let  $f$  be a regular function on  $\mathbb{B}$  with Taylor expansion

$$f(w) = \sum_{n=0}^{\infty} w^n a_n.$$

For each  $r \in (0, 1)$ , the *regular diameter* of the image of  $r\mathbb{B}$  under  $f$  is defined to be

$$(4.1) \quad \tilde{d}(f(r\mathbb{B})) := \max_{u, v \in \mathbb{B}} \max_{|w| \leq r} |f_u(w) - f_v(w)|.$$

The *regular diameter* of the image of  $\mathbb{B}$  under  $f$  is defined to be

$$(4.2) \quad \tilde{d}(f(\mathbb{B})) := \lim_{r \rightarrow 1^-} \tilde{d}(f(r\mathbb{B})).$$

Clearly,  $\tilde{d}(f(r\mathbb{B}))$  is an increasing function of  $r \in (0, 1)$ ; hence the limit in (4.2) always exists. Therefore,  $\tilde{d}(f(\mathbb{B}))$  is well-defined. Moreover, in view of the following maximum principle for regular functions,  $\tilde{d}(f(r\mathbb{B}))/2r$  is an increasing function of  $r \in (0, 1)$  as well (see (4.5) below).

**Theorem 4.3.** Let  $f : \Omega \rightarrow \mathbb{O}$  be a regular function on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$ . If there exist a  $I \in \mathbb{S}$  such that the restriction  $|f_I|$  of  $|f|$  to  $\Omega_I$  attains a local maximum at some point  $w_0 \in \Omega_I$ , then  $f$  is constant.

*Proof.* We can split  $f_I$  as

$$f(z) = F_1(z) + F_2(z)J + (F_3(z) + \overline{F_4(z)}J)K, \quad \forall z \in \Omega_I,$$

where  $F_k : \Omega_I \rightarrow \mathbb{C}_I$ ,  $k = 1, 2, 3, 4$ , are four holomorphic functions, and  $I, J, K$  enjoy the same property as in the proof of Lemma 3.2. Then the holomorphic mapping  $F : \Omega_I \rightarrow \mathbb{C}_I^4$  given by

$$F(z) := (F_1(z), F_2(z), F_3(z), F_4(z))$$

satisfies that

$$|F(z)|^2 = \sum_{k=1}^4 |F_k(z)|^2 = |f(z)|^2$$

for all  $z \in \Omega_I$ . By assumption,  $|F|$  attains a local maximum at the point  $w_0 \in \Omega_I$ . Thus from the maximum principle for holomorphic mappings (cf. [28, Theorem 2.8.3]) it immediately follows that  $F$  is constant on  $\Omega_I$ , and  $f$  is constant there as well, and in turn on  $\Omega$  by the identity principle.  $\square$

We also need the following results.

**Proposition 4.4.** *Let  $f$  be a regular function on  $\mathbb{B}$ . Then*

$$(4.3) \quad \text{diam } f(\mathbb{B}) \leq \tilde{d}(f(\mathbb{B})) \leq 2 \text{diam } f(\mathbb{B}).$$

**Lemma 4.5.** *Let  $g$  be a regular function on  $\mathbb{B}$  such that for each  $w \in \mathbb{B} \setminus \{0\}$ ,*

$$\langle I_w, g(w) \rangle = 0,$$

*where  $I_w = \text{Im } w / |\text{Im } w|$  is the pure imaginary unit identified by  $w$ . Then  $g$  is a real constant function.*

The proofs of Proposition 4.4 and Lemma 4.5 are completely the same as those of [12, Propositions 3.8 and 3.4], and so we omit them. Now we are ready to prove Theorem 1.4.

*Proof of Theorem 1.4.* The proof is partly the same as that of [12, Theorem 3.9], the main difference being that we use some extra technical treatments together with Theorem 1.1, instead of [12, Proposition 3.2], which is not enough for our purpose because of the non-associativity of octonions.

We first prove inequality (1.7). To this end, we take  $u, v \in \overline{\mathbb{B}}$  and consider the following auxiliary function

$$g_{u,v}(w) = \frac{1}{2}w^{-1}(f_u(w) - f_v(w)).$$

Then  $g_{u,v}$  is regular on  $\mathbb{B}$  with

$$(4.4) \quad g_{u,v}(0) = \frac{1}{2}(u - v)f'(0).$$

Applying the maximum principle (Theorem 4.3) to the regular function  $g_{u,v}$  yields that for each  $r \in (0, 1)$ , we can write

$$\max_{|w| \leq r} |g_{u,v}(w)| = \max_{|w| \leq r} \frac{|f_u(w) - f_v(w)|}{2|w|} = \frac{1}{2r} \max_{|w| \leq r} |f_u(w) - f_v(w)|,$$

which implies that

$$(4.5) \quad \frac{\tilde{d}(f(r\mathbb{B}))}{2r} = \frac{1}{2r} \max_{u,v \in \overline{\mathbb{B}}} \max_{|w| \leq r} |f_u(w) - f_v(w)| = \max_{u,v \in \overline{\mathbb{B}}} \max_{|w| \leq r} |g_{u,v}(w)|.$$

Therefore,  $\tilde{d}(f(r\mathbb{B}))/2r$  is an increasing function of  $r \in (0, 1)$  and so always not more than

$$\lim_{r \rightarrow 1^-} \frac{\tilde{d}(f(r\mathbb{B}))}{2r} = \frac{1}{2} \tilde{d}(f(\mathbb{B})) = 1.$$

This means that

$$(4.6) \quad \tilde{d}(f(r\mathbb{B})) \leq 2r$$

for each  $r \in (0, 1)$ , proving inequality (1.7). To prove inequality (1.8), consider the odd part of  $f$

$$f_{\text{odd}}(w) = \frac{1}{2}(f(w) - f(-w)),$$

which is regular on  $\mathbb{B}$  satisfying both  $f_{\text{odd}}(0) = 0$  and

$$|f_{\text{odd}}(w)| = \frac{1}{2}|f(w) - f(-w)| \leq \frac{1}{2} \tilde{d}(f(\mathbb{B})) = 1$$

for all  $w \in \mathbb{B}$ . Thus it follows from the Schwarz lemma that

$$(4.7) \quad |f'(0)| = |f'_{\text{odd}}(0)| \leq 1.$$

Now we come to prove the last assertion in the theorem. Obviously, if  $f(w) = f(0) + wf'(0)$  with  $|f'(0)| = 1$ , equality holds both in (1.7) and (1.8). Conversely, suppose that equality holds in (1.8), i.e.  $|f'(0)| = 1$ . Thus  $|f'_{\text{odd}}(0)| = 1$ , and then again by the Schwarz lemma,

$$f_{\text{odd}}(w) = wf'(0).$$

We next claim that in this case  $\tilde{d}(f(r\mathbb{B})) = 2r$  for each  $r \in (0, 1)$ . Indeed, from (4.4) and (4.5) it follows that for each  $r \in (0, 1)$ ,

$$\frac{\tilde{d}(f(r\mathbb{B}))}{2r} \geq \max_{u,v \in \mathbb{B}} |g_{u,v}(0)| = \frac{1}{2} \max_{u,v \in \mathbb{B}} |u - v| |f'(0)| = 1,$$

which together with (4.6) implies that

$$\tilde{d}(f(r\mathbb{B})) = 2r$$

for each  $r \in (0, 1)$ , as claimed.

Take  $\xi \in \mathbb{B} \setminus \{0\}$  with  $0 < |\xi| =: r < 1$  and set

$$(4.8) \quad h(w) = \frac{1}{2} (f(w) - f(-\xi)).$$

Thus  $h$  is regular on  $\mathbb{B}$  satisfying

$$h(\xi) = \frac{1}{2} (f(\xi) - f(-\xi)) = f_{\text{odd}}(\xi) = \xi f'(0).$$

Moreover, from the very definition (4.8) and Proposition 4.4 it follows that

$$(4.9) \quad \max_{|w| \leq r} |h(w)| = \frac{1}{2} \max_{|w| \leq r} |f(w) - f(-\xi)| \leq \frac{1}{2} \text{diam } f(r\mathbb{B}) \leq \frac{1}{2} \tilde{d}(f(r\mathbb{B})) = r = |h(\xi)|.$$

Therefore, the regular function  $h$  satisfies all the assumptions given in Theorem 1.1, and hence

$$(4.10) \quad \frac{\partial |h|}{\partial \xi}(\xi) = \bar{\xi} \left( h(\xi) \overline{h'(\xi)} + [\bar{\xi}, h(\xi) \overline{R_{\bar{\xi}} R_{\xi} h(\xi)}] + 2[\xi, h(\xi), R_{\bar{\xi}} R_{\xi} h(\xi)] \right) > 0.$$

In particular,

$$(4.11) \quad \begin{aligned} 0 &= \left\langle I_{\xi}, \bar{\xi} \left( h(\xi) \overline{h'(\xi)} + [\bar{\xi}, h(\xi) \overline{R_{\bar{\xi}} R_{\xi} h(\xi)}] + 2[\xi, h(\xi), R_{\bar{\xi}} R_{\xi} h(\xi)] \right) \right\rangle \\ &= \left\langle I_{\xi} \xi, h(\xi) \overline{h'(\xi)} + [\bar{\xi}, h(\xi) \overline{R_{\bar{\xi}} R_{\xi} h(\xi)}] + 2[\xi, h(\xi), R_{\bar{\xi}} R_{\xi} h(\xi)] \right\rangle \\ &= \left\langle I_{\xi} \xi, h(\xi) \overline{h'(\xi)} \right\rangle. \end{aligned}$$

Here  $I_{\xi} = \text{Im } \xi / |\text{Im } \xi|$  is the pure imaginary unit identified by  $\xi$ , the second equality follows from Lemma 2.2, and the last one follows from Lemma 2.3 and its proof.

Substituting the values of  $h(\xi)$  and  $h'(\xi)$  into the preceding inequalities yields that

$$\begin{aligned}
 0 &= \frac{1}{r^2} \left\langle \xi I_\xi, (\xi f'(0)) \overline{f'(\xi)} \right\rangle \\
 &= \frac{1}{r^2} \left\langle \xi I_\xi, [\xi, f'(0), \overline{f'(\xi)}] + \xi(f'(0) \overline{f'(\xi)}) \right\rangle \\
 (4.12) \quad &= \frac{1}{r^2} \left\langle \xi I_\xi, \xi(f'(0) \overline{f'(\xi)}) \right\rangle \\
 &= \left\langle I_\xi, f'(0) \overline{f'(\xi)} \right\rangle \\
 &= -\left\langle I_\xi, f'(\xi) \overline{f'(0)} \right\rangle.
 \end{aligned}$$

Here we have again used Lemmas 2.2 and 2.3. Therefore, for each  $\xi \in \mathbb{B} \setminus \{0\}$ ,

$$\begin{aligned}
 \left\langle I_\xi, f'(\xi) * \overline{f'(0)} \right\rangle &= \sum_{n=1}^{\infty} n \left\langle I_\xi, \xi^{n-1} (a_n \overline{f'(0)}) \right\rangle \\
 &= \sum_{n=1}^{\infty} n \left\langle I_\xi, (\xi^{n-1} a_n) \overline{f'(0)} - [\xi^{n-1}, a_n, \overline{f'(0)}] \right\rangle \\
 (4.13) \quad &= \sum_{n=1}^{\infty} n \left\langle I_\xi, (\xi^{n-1} a_n) \overline{f'(0)} \right\rangle \\
 &= \left\langle I_\xi, f'(\xi) \overline{f'(0)} \right\rangle \\
 &= 0.
 \end{aligned}$$

Thus by Lemma 4.5, the regular function

$$\xi \mapsto f'(\xi) * \overline{f'(0)}$$

must be a real constant function  $|f'(0)|^2 = 1$ , and hence  $f'(\xi) \equiv f'(0)$ . Consequently,  $f$  is of the desired form

$$f(w) = f(0) + w f'(0).$$

Now to complete the proof, it suffices to show that how equality in (1.7) for some  $r_0 \in (0, 1)$  implies equality in (1.8). This part is completely the same as that in the proof of [12, Theorem 3.9] and so we omit it.  $\square$

*Proof of Theorem 1.5.* The proof of this theorem is similar to that of Theorem 1.4. The only difference is that, instead of Theorem 1.1, we use the following simple observation. With the regular function  $h$  constructed in (4.8) and the function  $f$  in this theorem in mind, if  $|f'(0)| = 1$ , then  $f_{\text{odd}}(w) = w f'(0)$  and  $\text{diam } f(r\mathbb{B}_I) = 2r$  for each  $r \in (0, 1)$  and each  $I \in \mathbb{S}$ , and hence as in (4.9) we have

$$(4.14) \quad \max_{w \in r\mathbb{B}_{I_\xi}} |h(w)| = \frac{1}{2} \max_{w \in r\mathbb{B}_{I_\xi}} |f(w) - f(-\xi)| \leq \frac{1}{2} \text{diam } f(r\mathbb{B}_{I_\xi}) = r = |h(\xi)|.$$

Thus as in the proof of Theorem 1.1, we deduce that the directional derivative of  $|h|^2$  along the direction  $v_0 := I_\xi \xi \in T_\xi(\partial(r\mathbb{B}_{I_\xi}))$  at the point  $\xi \in \partial(r\mathbb{B}_{I_\xi})$  vanishes, i.e.

$$\frac{\partial |h|^2}{\partial v_0}(\xi) = 0.$$

This together with (3.14) with  $f$  replaced by  $h$  and  $v$  by  $v_0$  implies

$$(4.15) \quad \begin{aligned} 0 &= \left\langle I_\xi \xi, h(\xi) \overline{h'(\xi)} + [\bar{\xi}, h(\xi) \overline{R_{\bar{\xi}} R_\xi h(\xi)}] + 2[\xi, h(\xi), R_{\bar{\xi}} R_\xi h(\xi)] \right\rangle \\ &= \left\langle I_\xi \xi, h(\xi) \overline{h'(\xi)} \right\rangle, \end{aligned}$$

which is (4.11) except the first equality there and is sufficient for obtaining (4.13) and in turn the desired result. This completes the proof.  $\square$

*Proof of Theorem 1.6.* The argument is standard (cf. [4, p.149, Theorem 9.1]). Write  $a_k := f^{(k)}(0)/k!$ , so that

$$f(w) = \sum_{k=0}^{\infty} w^k a_k.$$

Fix a positive integer  $n$  and a  $I \in \mathbb{S}$ , consider the regular function on  $\mathbb{B}$  given by

$$(4.16) \quad g(w) = \sum_{k=0}^{\infty} w^k ((1 - e^{k\pi I/n}) a_k).$$

Notice that  $g_I(z) = f_I(z) - f_I(ze^{\pi I/n})$  holds for all  $z \in \mathbb{B}_I$ . Thus together with Lemma 2.2, the absolute and locally uniform convergence of the power series in (4.16) implies that for each  $r \in (0, 1)$ ,

$$(4.17) \quad \begin{aligned} d^2 &\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(re^{I\theta})|^2 d\theta \\ &= \frac{1}{2\pi} \sum_{k,l=0}^{\infty} r^{k+l} \int_{-\pi}^{\pi} \left\langle e^{kI\theta} ((1 - e^{k\pi I/n}) a_k), e^{lI\theta} ((1 - e^{l\pi I/n}) a_l) \right\rangle d\theta \\ &= \frac{1}{2\pi} \sum_{k,l=0}^{\infty} r^{k+l} \int_{-\pi}^{\pi} \left\langle e^{(k-l)I\theta} ((1 - e^{k\pi I/n}) a_k), (1 - e^{l\pi I/n}) a_l \right\rangle d\theta \\ &= \frac{1}{2\pi} \sum_{k,l=0}^{\infty} r^{k+l} \left\langle \left( \int_{-\pi}^{\pi} e^{(k-l)I\theta} d\theta \right) ((1 - e^{k\pi I/n}) a_k), (1 - e^{l\pi I/n}) a_l \right\rangle \\ &= \sum_{k=0}^{\infty} |1 - e^{k\pi I/n}|^2 |a_k|^2 r^{2k}. \end{aligned}$$

Thus by Lebesgue's monotone convergence theorem,

$$(4.18) \quad \sum_{k=0}^{\infty} |1 - e^{k\pi I/n}|^2 |a_k|^2 = \lim_{r \rightarrow 1^-} \sum_{k=0}^{\infty} |1 - e^{k\pi I/n}|^2 |a_k|^2 r^{2k} \leq d^2.$$

In particular,

$$|a_n| \leq \frac{d}{2},$$

which is precisely inequality (1.12).

If equality holds in (1.12) for some  $n_0$ , then (4.18) with  $n$  replaced by  $n_0$  implies that

$$(1 - e^{k\pi I/n_0}) a_k = 0$$

for all  $k \neq n_0$ . In particular,  $a_k = 0$  whenever  $k$  is not a multiple of  $n_0$ . Thus

$$(4.19) \quad f(w) = h(w^{n_0}),$$

where

$$h(w) := \sum_{k=0}^{\infty} w^k a_{kn_0},$$

which satisfies that

$$(4.20) \quad h'(0) = a_{n_0} \quad \text{and} \quad \text{Diam } h(\mathbb{B}) = \text{Diam } f(\mathbb{B}) = d.$$

Suppose that  $d > 0$ . By the very definition,

$$\widehat{d}(h(\mathbb{B})) \leq \text{diam } h(\mathbb{B}) = d.$$

This together with Theorem 1.5 implies

$$\frac{d}{2} = |a_{n_0}| = |h'(0)| \leq \frac{1}{2} \widehat{d}(h(\mathbb{B})) \leq \frac{d}{2}.$$

Consequently,

$$|h'(0)| = \frac{1}{2} \widehat{d}(h(\mathbb{B})) = \frac{d}{2}.$$

It immediately follows from Theorem 1.5 that

$$h(w) = h(0) + wh'(0),$$

which implies that  $f$  is of the desired form. The proof is complete.  $\square$

## 5. GEOMETRIC PROPERTIES OF OCTONIONIC SLICE REGULAR FUNCTIONS

In this section, we use some ideas developed in the proof of Theorem 1.1 to further investigate geometric properties of octonionic slice regular functions.

**5.1. The minimum principle and the open mapping theorem.** We begin with the following result, a special case of which has been used in (4.13).

**Proposition 5.1.** *Let  $f$  and  $g$  be two regular functions on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$ . Then the following two equalities hold:*

$$(5.1) \quad \left\langle I_w, f * g(w) \right\rangle = \left\langle I_w, f(w)g(f(w)^{-1}wf(w)) \right\rangle, \quad \forall w \in \Omega \setminus \mathcal{Z}_f,$$

and

$$(5.2) \quad |f^{-*}(w)| = \frac{1}{|f(f^c(w)^{-1}wf^c(w))|}, \quad \forall w \in \Omega \setminus \mathcal{Z}_{f^s}.$$

*Proof.* Let  $D \subseteq \mathbb{R}^2$  be a domain such that  $w = x + yI \in \Omega$  whenever  $(x, y) \in D$  and  $I \in \mathbb{S}$ . Since  $f$  and  $g$  are regular on  $\Omega$ , it follows from [19, Propositions 6 and 8] that there exist four smooth functions  $\alpha, \beta, \gamma, \delta : D \rightarrow \mathbb{O}$  with  $\alpha(x, y) = \alpha(x, -y)$ ,  $\beta(x, -y) = -\beta(x, y)$ ,  $\gamma(x, y) = \gamma(x, -y)$  and  $\delta(x, -y) = -\delta(x, y)$  such that

$$\begin{cases} \frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial y} = 0 \\ \frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} = 0 \end{cases}; \quad \begin{cases} \frac{\partial \gamma}{\partial x} - \frac{\partial \delta}{\partial y} = 0 \\ \frac{\partial \gamma}{\partial y} + \frac{\partial \delta}{\partial x} = 0 \end{cases}$$

and

$$(5.3) \quad f(x + yI) = \alpha(x, y) + I\beta(x, y), \quad g(x + yI) = \gamma(x, y) + I\delta(x, y)$$



for all  $(x, y) \in D$  and  $I \in \mathbb{S}$ . By Remark 2.10, the regular product  $f * g$  is defined equivalently by

$$(5.4) \quad \begin{aligned} (f * g)(x + yI) = & \left( \alpha(x, y)\gamma(x, y) - \beta(x, y)\delta(x, y) \right) \\ & + I \left( \alpha(x, y)\delta(x, y) + \beta(x, y)\gamma(x, y) \right). \end{aligned}$$

Fix an arbitrary point  $w = x_0 + y_0I_0$  with  $(x_0, y_0) \in D$  and  $I_0 \in \mathbb{S}$ . In the remaining argument, we will drop the coordinates  $(x_0, y_0)$  from  $\alpha(x_0, y_0)$ ,  $\beta(x_0, y_0)$ ,  $\gamma(x_0, y_0)$  and  $\delta(x_0, y_0)$  for the sake of simplicity. Assume that  $f(w) = \alpha + I_0\beta \neq 0$ . From Artin's theorem for alternative algebras (Theorem 2.1), we know that

$$[f(w), I_0, f(w)^{-1}] = 0$$

and hence  $(f(w)I_0)f(w)^{-1} = f(w)(I_0f(w)^{-1})$ , which belongs to  $\mathbb{S}$  and will be denoted by  $f(w)I_0f(w)^{-1}$  with no ambiguity. In view of (5.4), and Lemmas 2.2 and 2.3,

$$(5.5) \quad \begin{aligned} \langle I_0, f * g(w) \rangle &= \langle I_0, \alpha\gamma + I_0(\beta\gamma) \rangle + \langle I_0, I_0(\alpha\delta) - \beta\delta \rangle \\ &= \langle I_0, \alpha\gamma + (I_0\beta)\gamma \rangle + \langle 1, \alpha\delta \rangle - \langle I_0, \beta\delta \rangle \\ &= \langle I_0, (\alpha + I_0\beta)\gamma \rangle + \langle 1, \alpha\delta \rangle - \langle I_0, \beta\delta \rangle. \end{aligned}$$

Next we claim that

$$\langle I_0, (\alpha + I_0\beta) \left( ((\alpha + I_0\beta)^{-1}I_0(\alpha + I_0\beta))\delta \right) \rangle = \langle 1, \alpha\delta \rangle - \langle I_0, \beta\delta \rangle,$$

from which together with (5.5) it will immediately follow that

$$\begin{aligned} \langle I_0, f * g(w) \rangle &= \langle I_0, (\alpha + I_0\beta)\gamma \rangle + \langle I_0, (\alpha + I_0\beta) \left( ((\alpha + I_0\beta)^{-1}I_0(\alpha + I_0\beta))\delta \right) \rangle \\ &= \langle I_0, f(w) \left( \gamma + (f(w)^{-1}I_0f(w))\delta \right) \rangle \\ &= \langle I_0, f(w)g(f(w)^{-1}wf(w)) \rangle, \end{aligned}$$

since  $f(w)^{-1}wf(w) \in \mathbb{S}_w$  and  $g(f(w)^{-1}wf(w)) = \gamma + (f(w)^{-1}I_0f(w))\delta$ .

Thanks to (2.5), and Lemmas 2.2 and 2.3, the preceding claim can be proved as follows:

$$\begin{aligned} & \langle I_0, (\alpha + I_0\beta) \left( ((\alpha + I_0\beta)^{-1}I_0(\alpha + I_0\beta))\delta \right) \rangle \\ &= \langle \overline{(\alpha + I_0\beta)}I_0, \left( (\alpha + I_0\beta)^{-1}I_0(\alpha + I_0\beta) \right)\delta \rangle \\ &= |\alpha + I_0\beta|^2 \langle (\alpha + I_0\beta)^{-1}I_0, \left( (\alpha + I_0\beta)^{-1}I_0(\alpha + I_0\beta) \right)\delta \rangle \\ &= |\alpha + I_0\beta|^2 \langle (\alpha + I_0\beta)^{-1}I_0, [(\alpha + I_0\beta)^{-1}I_0, \alpha + I_0\beta, \delta] \\ & \quad + \left( (\alpha + I_0\beta)^{-1}I_0 \right) \left( (\alpha + I_0\beta)\delta \right) \rangle \\ &= |\alpha + I_0\beta|^2 \langle (\alpha + I_0\beta)^{-1}I_0, \left( (\alpha + I_0\beta)^{-1}I_0 \right) \left( (\alpha + I_0\beta)\delta \right) \rangle \\ &= \langle 1, (\alpha + I_0\beta)\delta \rangle \\ &= \langle 1, \alpha\delta \rangle + \langle 1, (I_0\beta)\delta \rangle \\ &= \langle 1, \alpha\delta \rangle + \langle 1, I_0(\beta\delta) \rangle \\ &= \langle 1, \alpha\delta \rangle - \langle I_0, \beta\delta \rangle. \end{aligned}$$

Now the proof of equality (5.1) is complete and it remains to prove equality (5.2). To this end, we further assume that  $f^s(w) \neq 0$ , i.e  $w \in \Omega \setminus \mathcal{Z}_{fs}$ . By [19, Corollary 19],

$$(5.6) \quad \mathcal{Z}_{fs} = \bigcup_{u \in \mathcal{Z}_f} \mathbb{S}_u = \bigcup_{u \in \mathcal{Z}_{fc}} \mathbb{S}_u.$$

Therefore, the restrictions  $f|_{\mathbb{S}_w}$  and  $f^c|_{\mathbb{S}_w}$  never vanish so that  $f^c(w)^{-1}wf^c(w)$  makes sense and

$$(5.7) \quad \tilde{w} := f^c(w)^{-1}wf^c(w) = x_0 + y_0f^c(w)^{-1}I_0f^c(w) \in \mathbb{S}_w \subseteq \Omega \setminus \mathcal{Z}_{fs}.$$

Furthermore, it follows from (5.3) and Remark 2.13 that

$$f^c(x + yI) = \bar{\alpha}(x, y) + I\bar{\beta}(x, y)$$

for all  $(x, y) \in D$  and  $I \in \mathbb{S}$ , from which we deduce that

$$(5.8) \quad \begin{aligned} f^{-*}(w) &= \frac{1}{f^s(w)}f^c(w) \\ &= \left(|\alpha|^2 - |\beta|^2 + 2I_0\langle\alpha, \beta\rangle\right)^{-1}(\bar{\alpha} + I_0\bar{\beta}) \end{aligned}$$

so that

$$|f^{-*}(w)|^2 = \frac{|\alpha|^2 + |\beta|^2 + 2\langle\bar{\alpha}\beta, I_0\rangle}{(|\alpha|^2 - |\beta|^2)^2 + 4\langle\alpha, \beta\rangle^2}.$$

Now to conclude the proof, it suffices to prove that

$$(5.9) \quad |f(\tilde{w})|^2 = \frac{(|\alpha|^2 - |\beta|^2)^2 + 4\langle\alpha, \beta\rangle^2}{|\alpha|^2 + |\beta|^2 + 2\langle\bar{\alpha}\beta, I_0\rangle}.$$

Since

$$f(\tilde{w}) = \alpha + (f^c(w)^{-1}I_0f^c(w))\beta = \alpha + \left((\bar{\alpha} + I_0\bar{\beta})^{-1}I_0(\bar{\alpha} + I_0\bar{\beta})\right)\beta,$$

it follows that

$$(5.10) \quad \begin{aligned} |f(\tilde{w})|^2 &= |\alpha|^2 + |\beta|^2 + 2\left\langle\alpha, \left((\bar{\alpha} + I_0\bar{\beta})^{-1}I_0(\bar{\alpha} + I_0\bar{\beta})\right)\beta\right\rangle \\ &= |\alpha|^2 + |\beta|^2 + 2|\alpha - \beta I_0|^{-2}\left\langle\alpha, \left((\alpha - \beta I_0)I_0(\bar{\alpha} + I_0\bar{\beta})\right)\beta\right\rangle. \end{aligned}$$

We next claim that

$$\left\langle\alpha, \left((\alpha - \beta I_0)I_0(\bar{\alpha} + I_0\bar{\beta})\right)\beta\right\rangle = -(|\alpha|^2 + |\beta|^2)\langle\bar{\alpha}\beta, I_0\rangle + 2\langle\alpha, \beta\rangle^2 - 2|\alpha|^2|\beta|^2,$$

from which (5.9) will immediately follow and the proof will be concluded. A direct computation shows that the left-hand side of the preceding equality is exactly

$$(5.11) \quad \begin{aligned} &\left\langle\alpha, \left(\alpha I_0\bar{\alpha} + \beta I_0\bar{\beta} + \beta\bar{\alpha} - \alpha\bar{\beta}\right)\beta\right\rangle \\ &= \left\langle\alpha, \alpha(I_0\bar{\alpha})\beta\right\rangle + [\alpha, I_0\bar{\alpha}, \beta] + |\beta|^2\langle\alpha, \beta I_0\rangle + \langle\alpha\bar{\beta}, \beta\bar{\alpha}\rangle - |\alpha|^2|\beta|^2 \\ &= |\alpha|^2\langle 1, (I_0\bar{\alpha})\beta\rangle + |\beta|^2\langle\bar{\beta}\alpha, I_0\rangle + \langle\alpha\bar{\beta}, 2\langle\alpha, \beta\rangle - \alpha\bar{\beta}\rangle - |\alpha|^2|\beta|^2 \\ &= |\alpha|^2\langle 1, I_0(\bar{\alpha}\beta)\rangle - |\beta|^2\langle\bar{\alpha}\beta, I_0\rangle + 2\langle\alpha, \beta\rangle^2 - 2|\alpha|^2|\beta|^2 \\ &= -(|\alpha|^2 + |\beta|^2)\langle\bar{\alpha}\beta, I_0\rangle + 2\langle\alpha, \beta\rangle^2 - 2|\alpha|^2|\beta|^2, \end{aligned}$$

which is precisely the right-hand side as desired.  $\square$

Now we come to prove the following weak version of the minimum principle:

**Theorem 5.2.** *Let  $f : \Omega \rightarrow \mathbb{O}$  be a regular function on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$ . If  $|f|$  attains a local minimum at some point  $w_0 \in \Omega \cap \mathbb{R}$ , then either  $f(w_0) = 0$  or  $f$  is constant.*

We give two proofs of the theorem, which seem useful for other purposes.

*The first proof of Theorem 5.2.* We use a variational argument similar to the proof of Theorem 1.1. Without loss of generality, we may assume that  $0 \in \Omega$  and  $w_0 = 0$ . Suppose by contradiction that  $f(0) \neq 0$  or  $f$  is not constant. For each  $\xi \in \partial\mathbb{B}$ , we consider the function

$$\psi_\xi(t) := |f(t\xi)|^2$$

defined on some interval  $(-\varepsilon, \varepsilon)$  with  $\varepsilon > 0$  sufficiently small. By assumption,  $\psi_\xi(0)$  is a minimum of  $\psi_\xi$ , and hence

$$2\langle f(0)\overline{f'(0)}, \xi \rangle = 2\langle f(0), \xi f'(0) \rangle = 2 \left\langle f(t\xi), \frac{d}{dt}f(t\xi) \right\rangle \Big|_{t=0} = \psi'_\xi(0) = 0$$

for all  $\xi \in \partial\mathbb{B}$ . Therefore,  $f(0)\overline{f'(0)} = 0$ , i.e.  $f'(0) = 0$ . Since  $f$  is not constant, there must exist a positive integer  $n_0 \geq 2$  such that  $f^{(n_0)}(0) \neq 0$ , but  $f'(0) = f''(0) = \dots = f^{(n_0-1)}(0) = 0$ . Thus it holds that

$$\begin{aligned} \psi_\xi(t) &= \left| f(0) + t^{n_0} \xi^{n_0} f^{(n_0)}(0)/n_0! + o(t^{n_0}) \right|^2 \\ (5.12) \quad &= |f(0)|^2 + 2t^{n_0} \left\langle f(0)\overline{f^{(n_0)}(0)}/n_0!, \xi^{n_0} \right\rangle + o(t^{n_0}) \end{aligned}$$

as  $t \rightarrow 0$ . Then

$$\psi'_\xi(0) = \psi''_\xi(0) = \dots = \psi_\xi^{(n_0-1)}(0) = 0$$

and

$$\psi_\xi^{(n_0)}(0) = 2 \left\langle f(0)\overline{f^{(n_0)}(0)}, \xi^{n_0} \right\rangle.$$

If  $n_0$  is even, then the minimality of  $\psi_\xi(0)$  implies that  $\psi_\xi^{(n_0)}(0) \geq 0$ , i.e.

$$\left\langle f(0)\overline{f^{(n_0)}(0)}, \xi^{n_0} \right\rangle \geq 0$$

for all  $\xi \in \partial\mathbb{B}$ . This is possible only if  $f^{(n_0)}(0) = 0$ , which is a contradiction. If  $n_0$  is odd, then  $\psi_\xi(0) = 0$  for all  $\xi \in \partial\mathbb{B}$ . This also implies that  $f^{(n_0)}(0) = 0$ , giving a contradiction as well. The proof is complete.  $\square$

*The second proof of Theorem 5.2.* Without loss of generality, we may assume that  $0 \in \Omega$  and  $w_0 = 0$ . We further assume that  $f(0) \neq 0$  and use Proposition 5.1 to deduce that  $f(0)^{-1}$  is a local maximum of  $f^{-*}$ , and thus it follows from the maximum principle (Theorem 4.3) that  $f^{-*}$  is constant and so is  $f$ .

First in view of Remark 2.14, the symmetrization  $f^s$  is slice preserving, i.e.  $f^s(w)$  and  $w$  always lie in a same complex plane and thus commute. Then by equality (5.2) with  $f$  replaced by  $f^{-*}$  and Artin's theorem for alternative algebras (Theorem 2.1), the following equality

$$(5.13) \quad |f(w)| = \frac{1}{|f^{-*}(f(w)^{-1}wf(w))|}$$

holds for all  $w \in \Omega \setminus \mathcal{Z}_{f^s}$ . We next consider the real differential at the point  $w_0 = 0$  of the function given by

$$g(w) := f(w)^{-1}wf(w) = |f(w)|^{-2}\overline{f(w)}wf(w)$$

on  $\Omega \setminus \mathcal{Z}_{fs} \ni 0$ . Since  $|f|^{-2}$  attains a local maximum  $|f(0)|^{-2}$  at 0, its directional derivative along every direction at 0 always vanishes. Now a simple calculation gives that for all  $v \in \partial\mathbb{B}$ ,

$$\frac{\partial g}{\partial v}(0) = |f(0)|^{-2} \overline{f(0)} v f(0) = f(0)^{-1} v f(0) \neq 0.$$

This means exactly that the real differential  $dg_0$  of  $g$  at 0 is invertible. Thus in view of the inverse mapping theorem,  $g$  is a diffeomorphism from  $B(0, r_1)$  onto  $B(0, r_2)$ , where  $r_1$  and  $r_2$  are two small positive numbers. Therefore,  $f(0)^{-1}$  is a local maximum of  $f^{-*}$  in virtue of equality (5.13), and the desired result immediately follows.  $\square$

*Remark 5.3.* Now a fairly natural question arises of whether the restriction of  $w_0$  belonging to  $\Omega \cap \mathbb{R}$  in Theorem 5.2 is superfluous. The point in the second proof of Theorem 5.2 is to prove that the real differential of  $g$  at the point  $w = w_0$ , which is the minimum point of  $|f|$  such that  $f(w_0) \neq 0$ , is non-degenerate. In the general case that  $w_0 \in \Omega \setminus \mathbb{R}$ , the author does not know whether the preceding fact necessarily holds, and merely know that the rank of the real differential of  $g$  at the point  $w = w_0$  is greater than or equal to 4. If this were the case, the general minimum principle would immediately follow (Another possible approach to the general minimum principle is an argument analogous to that in the first proof of Theorem 5.2, by means of the so-called spherical power series expansion [22, Theorem 5.4] for octonionic slice regular functions), and in turn would imply the open mapping theorem analogous to [14, Theorem 7.7]. However, we have the following result, which corresponds to [14, Theorem 7.4].

**Theorem 5.4.** *Let  $f : \Omega \rightarrow \mathbb{O}$  be a nonconstant regular function on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$ . If  $U$  is a symmetric open subset of  $\Omega$ , then  $f(U)$  is open. In particular,  $f(\Omega)$  is open.*

For each  $\alpha \in \mathbb{O}$  and each  $\delta > 0$ , we denote by  $V_{\alpha\delta}$  the symmetric open subset of  $\mathbb{O}$  given by

$$V_{\alpha\delta} := \{w \in \mathbb{O} : d(w, \mathbb{S}_\alpha) < \delta\},$$

where  $d(\cdot, \cdot)$  is the Euclidean distance function on  $\mathbb{O}$ . For each non-identically vanishing regular function  $f$  on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$ , we define  $\mathcal{L}_f$  to be the slice preserving function on  $\Omega \setminus \mathcal{Z}_{fs}$  given by

$$\mathcal{L}_f(w) = \frac{(f^s)'(w)}{f^s(w)},$$

which plays the role of the logarithmic derivatives of holomorphic functions of one complex variable. Before presenting a proof of Theorem 5.4, we first prove the following

**Proposition 5.5.** *Let  $f : \Omega \rightarrow \mathbb{O}$  be a non-identically vanishing regular function on a symmetric slice domain  $\Omega \subseteq \mathbb{O}$ . Let  $\alpha \in \Omega$  and  $\delta > 0$  be such that  $V_{\alpha\delta} \subset\subset \Omega$  and  $f^s$  never vanishes on  $\partial V_{\alpha\delta}$ .*

(i) *If  $\alpha$  is a zero of  $f$ , then the value of the following integral*

$$\frac{1}{2\pi I} \int_{\partial V_{\alpha\delta} \cap \mathbb{C}_I} \mathcal{L}_f(z) dz$$

*is a positive integer depending only on  $\alpha$ ,  $\delta$  and independent of  $I \in \mathbb{S}$ ;*

(ii) *If*

$$\frac{1}{2\pi I} \int_{\partial V_{\alpha\delta} \cap \mathbb{C}_I} \mathcal{L}_f(z) dz > 0,$$

*then  $f$  must have a zero on  $V_{\alpha\delta}$ .*

*Proof.* The result follows immediately from the complex argument principle applied to the slice preserving function  $f^s$  (see Remark 2.14), together with [19, Corollary 19].  $\square$

Now we come to prove Theorem 5.4.

*Proof of Theorem 5.4.* Let  $U \subseteq \Omega$  be as described. Fix an arbitrary point  $\omega_0 \in f(U)$ . Choose one point  $\alpha \in U$  with  $f(\alpha) = \omega_0$ , so that  $f - \omega_0$  has a zero on  $\mathbb{S}_\alpha \subseteq U$ . Since  $f$  is nonconstant, we may choose a  $\delta > 0$  such that  $V_{\alpha\delta} \subset\subset U$  and  $f(w) - \omega_0 \neq 0$  for all  $w \in \overline{V}_{\alpha\delta} \setminus \mathbb{S}_\alpha$ . Let  $\varepsilon > 0$  be such that

$$\min_{w \in \partial V_{\alpha\delta}} |f(w) - \omega_0| \geq 2\varepsilon.$$

For each  $\omega \in B(\omega_0, \varepsilon)$ , we have

$$\min_{w \in \partial V_{\alpha\delta}} |f(w) - \omega| \geq \min_{w \in \partial V_{\alpha\delta}} |f(w) - \omega_0| - |\omega - \omega_0| > \varepsilon.$$

This together with [19, Corollary 19] implies that for each  $\omega \in B(\omega_0, \varepsilon)$ , the symmetrization  $(f - \omega)^s$  of the regular function  $f - \omega$  never vanishes on the boundary  $\partial V_{\alpha\delta}$  so that the following integral

$$\frac{1}{2\pi I} \int_{\partial V_{\alpha\delta} \cap \mathbb{C}_I} \mathcal{L}_{f-\omega}(z) dz$$

is well-defined and thus determines a function of  $\omega \in B(\omega_0, \varepsilon)$ , depending only on  $\alpha, \delta$  and independent of  $I \in \mathbb{S}$ . This function is obviously continuous and takes values in  $\mathbb{N}$  by Proposition 5.5 (i), and hence equals identically to a positive integer, since

$$\frac{1}{2\pi I} \int_{\partial V_{\alpha\delta} \cap \mathbb{C}_I} \mathcal{L}_{f-\omega_0}(z) dz \geq 1.$$

Thus it follows from Proposition 5.5 (ii) that for each  $\omega \in B(\omega_0, \varepsilon)$ ,  $f - \omega$  must have a zero on  $V_{\alpha\delta}$ . In other words,  $B(\omega_0, \varepsilon) \subseteq f(V_{\alpha\delta}) \subseteq f(U)$ . Since  $\omega_0 \in f(U)$  is arbitrarily chosen, we conclude the proof.  $\square$

*Remark 5.6.* (i) It is easy to see that Theorem 5.4 also implies Theorem 5.2.

(ii) Thanks to Theorem 5.4 together with the standard slice technique, one can also prove the octonionic versions of the classical Carathéodory and Borel-Carathéodory theorem with an approach different from and simpler than the one given in [34], and in turn the Bohr theorem. We leave the details to the interested reader.

## 5.2. The growth, distortion and covering theorems.

**Theorem 5.7.** *Let  $f$  be a regular function on  $\mathbb{B}$  such that its restriction  $f_I$  to  $\mathbb{B}_I$  is injective and  $f(\mathbb{B}_I) \subseteq \mathbb{C}_I$  for some  $I \in \mathbb{S}$ . If  $f(0) = 0$  and  $f'(0) = 1$ , then for all  $w \in \mathbb{B}$ , the following inequalities hold:*

$$(5.14) \quad \frac{|w|}{(1 + |w|)^2} \leq |f(w)| \leq \frac{|w|}{(1 - |w|)^2};$$

$$(5.15) \quad \frac{1 - |w|}{(1 + |w|)^3} \leq |f'(w)| \leq \frac{1 + |w|}{(1 - |w|)^3};$$

$$(5.16) \quad \frac{1 - |w|}{1 + |w|} \leq |wf'(w) * f^{-*}(w)| \leq \frac{1 + |w|}{1 - |w|}.$$

Moreover, equality holds for one of these six inequalities at some point  $w_0 \in \mathbb{B} \setminus \{0\}$  if and only if  $f$  is of the form

$$f(w) = w(1 - we^{I\theta})^{-*2}$$

with some  $\theta \in \mathbb{R}$ .

The proof of the preceding theorem is similar to the one in [36, Theorem 3.5]. The only difference is that we use Proposition 3.5, instead of [36, Lemma 3.2]. So we omit the details.

Now we digress to the Koebe one-quarter theorem (Theorem 1.9) for octonionic regular functions on the open unit ball  $\mathbb{B} \subset \mathbb{O}$ . We begin with the following simple result.

**Proposition 5.8.** *Let  $\Omega \subseteq \mathbb{O}$  be a bounded domain and  $f : \Omega \rightarrow \mathbb{O}$  a continuous function such that  $f(\Omega)$  is open in  $\mathbb{O}$ . Let  $\alpha \in \Omega$  be a point such that*

$$(5.17) \quad \rho := \liminf_{w \rightarrow \partial\Omega} |f(w) - f(\alpha)| > 0.$$

*Then  $B(f(\alpha), \rho) \subseteq f(\Omega)$ .*

*Proof.* For each point  $\omega$  on the boundary  $\partial f(\Omega)$  of  $f(\Omega)$ , there is a sequence  $\{w_n\}_{n=1}^\infty$  in  $\Omega$  such that  $\lim_{n \rightarrow \infty} f(w_n) = \omega$ . Since  $\overline{\Omega}$  is compact, we may assume that  $\{w_n\}_{n=1}^\infty$  converges to a point, say  $w_\infty \in \overline{\Omega}$ . If  $w_\infty \in \Omega$ , then, by the continuity of  $f$ ,  $\omega = f(w_\infty) \in f(\Omega)$ , which contradicts the openness of  $f(\Omega)$ . Therefore,  $w_\infty \in \partial\Omega$ . This together with (5.17) implies that

$$|\omega - f(\alpha)| = \lim_{n \rightarrow \infty} |f(w_n) - f(\alpha)| \geq \liminf_{w \rightarrow \partial\Omega} |f(w) - f(\alpha)| = \rho > 0.$$

Therefore, the boundary  $\partial f(\Omega)$  of the open set  $f(\Omega)$  lies outside of the ball  $B(f(\alpha), \rho)$ . Consequently,  $f(\Omega)$  must contain the ball  $B(f(\alpha), \rho)$ .  $\square$

*Proof of Theorem 1.9.* In view of Theorem 5.4,  $f(\mathbb{B})$  is open in  $\mathbb{O}$ . Since  $f(0) = 0$ , the desired result immediately follows from Proposition 5.8 and the first inequality in (5.14).  $\square$

## 6. A NEW AND SHARP BOUNDARY SCHWARZ LEMMA FOR QUATERNIONIC SLICE REGULAR FUNCTIONS

In this section, we turn our attention to quaternionic slice regular functions. In this special setting, with a completely new approach, we can strengthen a result first proved in [37] by the author and Ren, analogous to Theorem 1.1. Our quaternionic boundary Schwarz lemma with optimal estimate involves a Lie bracket, improves considerably a well-known Osserman type estimate and provides additionally all the extremal functions.

**6.1. Quaternionic slice regular functions.** Let  $\mathbb{H}$  denote the non-commutative, associative, real algebra of quaternions with standard basis  $\{1, i, j, k\}$ , subject to the multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1.$$

Let  $\langle \cdot, \cdot \rangle$  denote the standard inner product on  $\mathbb{H} \cong \mathbb{R}^4$ , i.e.

$$\langle p, q \rangle = \operatorname{Re}(p\bar{q}) = \sum_{n=0}^3 x_n y_n$$

for any  $p = x_0 + x_1 i + x_2 j + x_3 k$ ,  $q = y_0 + y_1 i + y_2 j + y_3 k \in \mathbb{H}$ .

We shall consider the slice regular functions defined on domains in quaternions  $\mathbb{H}$  with values in  $\mathbb{H}$ . To introduce the theory of quaternionic slice regular functions, we will denote by  $\mathbb{S}$  the unit 2-sphere of purely imaginary quaternions, i.e.

$$\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}.$$

For a given element  $\xi \in \mathbb{H}$ , we denote by  $\mathbb{S}_\xi$  the associated 2-sphere (reduces to the point  $\xi$  when  $\xi$  is real):

$$\mathbb{S}_\xi := \{q\xi q^{-1} : q \in \mathbb{H} \setminus \{0\}\}.$$

Recall that two quaternions belong to the same sphere if and only if they have the same modulus and the same real part. For every  $I \in \mathbb{S}$  we will denote by  $\mathbb{C}_I$  the plane  $\mathbb{R} \oplus I\mathbb{R}$ , isomorphic to  $\mathbb{C}$ , and, if  $\Omega \subseteq \mathbb{H}$ , by  $\Omega_I$  the intersection  $\Omega \cap \mathbb{C}_I$ . Also, we will denote by  $B(0, R)$  the Euclidean open ball of radius  $R$  centred at the origin, i.e.

$$B(0, R) = \{q \in \mathbb{H} : |q| < R\}.$$

For simplicity, we denote by  $\mathbb{B}$  the ball  $B(0, 1)$ .

We can now recall the definition of slice regularity.

**Definition 6.1.** Let  $\Omega$  be a domain in  $\mathbb{H}$ . A function  $f : \Omega \rightarrow \mathbb{H}$  is called (left) *slice regular* if, for all  $I \in \mathbb{S}$ , its restriction  $f_I$  to  $\Omega_I$  is *holomorphic*, i.e., it has continuous partial derivatives and satisfies

$$\bar{\partial}_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) = 0$$

for all  $x + yI \in \Omega_I$ .

The notions of slice domain, of symmetric slice domain and of slice derivative are similar to those already given in Section 2. Moreover, the corresponding results hold of course for quaternionic slice regular functions, such as the splitting lemma, the representation formula, the power series expansion and so on. The regular product, regular conjugate, symmetrization and regular reciprocal of quaternionic slice regular functions can also be defined in an analogous way. To have a more complete insight on the theory, we refer the reader to the monograph [14].

For our later purpose, we need to recall some results. The first one clarifies a nice connection between the regular product and the usual pointwise one (see [5, 13]):

**Proposition 6.2.** *Let  $f$  and  $g$  be slice regular on  $B = B(0, R)$ . Then for all  $q \in B$ ,*

$$f * g(q) = \begin{cases} f(q)g(f(q)^{-1}qf(q)) & \text{if } f(q) \neq 0; \\ 0 & \text{if } f(q) = 0. \end{cases}$$

The second one shows that the regular quotient is nicely related to the pointwise quotient (see [43, 45]):

**Proposition 6.3.** *Let  $f$  and  $g$  be slice regular on  $B = B(0, R)$ . Then for all  $q \in B \setminus \mathcal{Z}_{fs}$ ,*

$$f^{-*} * g(q) = f(T_f(q))^{-1}g(T_f(q)),$$

where  $T_f : B \setminus \mathcal{Z}_{fs} \rightarrow B \setminus \mathcal{Z}_{fs}$  is defined by  $T_f(q) = f^c(q)^{-1}qf^c(q)$ . Furthermore,  $T_f$  and  $T_{f^c}$  are mutual inverses so that  $T_f$  is a diffeomorphism.

For any two numbers  $x_0, y_0 \in \mathbb{R}$  and each  $R > 0$ , we denote by  $U(x_0 + y_0\mathbb{S}, R)$  the symmetric open subset of  $\mathbb{H}$  given by

$$U(x_0 + y_0\mathbb{S}, R) := \{q \in \mathbb{H} : |(q - x_0)^2 + y_0^2| < R^2\}.$$

The third one was the so-called spherical series expansion proved in [46] for slice regular functions; see Theorems 4.1 and 6.1 there for more details.

**Theorem 6.4.** *Let  $f$  be a slice regular function on a symmetric slice domain  $\Omega$ , and let  $q_0 = x_0 + Iy_0 \in U(x_0 + y_0\mathbb{S}, R) \subseteq \Omega$ . Then there exists  $\{A_n\}_{n \in \mathbb{N}} \subset \mathbb{H}$  such that*

$$(6.1) \quad f(q) = \sum_{n=0}^{\infty} ((q - x_0)^2 + y_0^2)^n (A_{2n} + (q - q_0)A_{2n+1})$$

for all  $q \in U(x_0 + y_0\mathbb{S}, R)$ .

As a consequence of Theorem 6.4, we obtain that for all  $v \in \mathbb{H}$  with  $|v| = 1$  the directional derivative of  $f$  along  $v$  at a point  $q_0$  is given by

$$\frac{\partial f}{\partial v}(q_0) = \lim_{t \rightarrow 0} \frac{f(q_0 + tv) - f(q_0)}{t} = vA_1 + (q_0v - v\bar{q}_0)A_2,$$

where

$$A_1 = R_{q_0}f(\bar{q}_0) = \partial_s f(q_0) \quad \text{and} \quad A_2 = R_{\bar{q}_0}R_{q_0}f(q_0)$$

are obtained in the same manner as in (3.1) and (3.2). In particular, there holds that

$$f'(q_0) = R_{q_0}f(q_0) = A_1 + 2\operatorname{Im}(q_0)A_2.$$

**6.2. Formulation and proof of quaternionic boundary Schwarz lemma.** Our subsequent argument involves the so-called slice regular Möbius transformations of  $\mathbb{B}$  onto  $\mathbb{B}$ , which are slice regular functions  $f$  on  $\mathbb{B}$  given by

$$f(q) = (1 - q\bar{u})^{-*} * (q - u)v$$

with  $u \in \mathbb{B}$  and  $v \in \partial\mathbb{B}$  (see [44, Corollary 7.2]; also [14, Corollary 9.17]). It is also useful to recall the quaternionic version of the classical Julia lemma (see [37, Theorem 1]):

**Theorem 6.5.** *Let  $f$  be a slice regular self-mapping of the open unit ball  $\mathbb{B}$  and let  $\xi \in \partial\mathbb{B}$ . Suppose that there exists a sequence  $\{q_n\}_{n \in \mathbb{N}} \subset \mathbb{B}$  converging to  $\xi$  as  $n$  tends to  $\infty$ , such that the limits*

$$\alpha := \lim_{n \rightarrow \infty} \frac{1 - |f(q_n)|}{1 - |q_n|}$$

and

$$\eta := \lim_{n \rightarrow \infty} f(q_n)$$

exist (finitely). Then  $\alpha > 0$  and the inequality

$$(6.2) \quad \operatorname{Re}\left((1 - f(q)\bar{\eta})^{-*} * (1 + f(q)\bar{\eta})\right) \geq \frac{1}{\alpha} \operatorname{Re}\left((1 - q\bar{\xi})^{-*} * (1 + q\bar{\xi})\right)$$

holds throughout the open unit ball  $\mathbb{B}$  and is strict except for slice regular Möbius transformations of  $\mathbb{B}$ .

Inequality (6.2) will be called Julia's inequality for the convenience of referring back to it.

Now we state and prove the main result of this section. The proof is based on Theorem 6.5, instead of a Lindelöf type inequality proved in [37, Proposition 3].

**Theorem 6.6.** *Let  $\xi \in \partial\mathbb{B}$  and  $f$  be a slice regular function on  $\mathbb{B} \cup \mathbb{S}_\xi$  such that  $f(\mathbb{B}) \subseteq \mathbb{B}$  and  $f(\xi) \in \partial\mathbb{B}$ . Denote by  $\delta$  the quantity*

$$\bar{\xi}\left(f(\xi)\overline{f'(\xi)} + [\bar{\xi}, f(\xi)R_{\bar{\xi}}R_{\xi}f(\xi)]\right).$$

Then



(i) the following sharp estimate holds:

$$(6.3) \quad \delta \geq \frac{2}{\mathcal{S} + \frac{1 - |f(0)|^2}{|f(\xi) - f(0)|^2}},$$

where

$$(6.4) \quad \mathcal{S} := \operatorname{Re} \left( f'(0) (f(\xi) - f(0))^{-1} \xi (1 - f(0) \overline{f(\xi)})^{-1} \right),$$

and

$$R_\xi f(q) := (q - \xi)^{-*} * (f(q) - f(\xi)).$$

Equality in inequality (6.3) holds if and only if  $f$  is of the form

$$(6.5) \quad f(q) = \left( 1 - q(1 - qa\bar{\eta})^{-*} * (q\bar{\eta} - a) \overline{f(0)v} \right)^{-*} * \left( f(0) - q(1 - qa\bar{\eta})^{-*} * (q\bar{\eta} - a) \bar{v} \right),$$

where

$$a \in [-1, 1), \quad v = (f(0) - f(\xi))^{-1} \xi (1 - f(\xi) \overline{f(0)}) \in \partial \mathbb{B},$$

and

$$\eta = (1 - f(\xi) \overline{f(0)})^{-1} \xi (1 - f(\xi) \overline{f(0)}) \in \partial \mathbb{B}.$$

Moreover, it holds that

$$(6.6) \quad \left\langle f(t\xi), f(\xi) \right\rangle \geq \frac{(\delta + 1)t - (\delta - 1)}{(\delta + 1) - (\delta - 1)t}, \quad \forall t \in (-1, 1),$$

with equality for some  $t_0 \in (-1, 1)$  if and only if

$$(6.7) \quad f(q) = \left( q(\delta - 1) - \xi(\delta + 1) \right)^{-*} * \left( \xi(\delta - 1) - q(\delta + 1) \right) f(\xi).$$

(ii) if further

$$f^{(k)}(0) = 0, \quad \forall k = 0, 1, \dots, n - 1$$

for some  $n \in \mathbb{N}$ , then

$$\delta \geq n + \frac{2}{\mathcal{T} + \frac{1 - |f^{(n)}(0)/n!|^2}{|f(\xi) - \xi^n f^{(n)}(0)/n!|^2}},$$

where

$$\mathcal{T} := \operatorname{Re} \left( \frac{f^{(n+1)}(0)}{(n+1)!} \left( \xi^{-n} f(\xi) - f^{(n)}(0)/n! \right) \xi \left( 1 - f^{(n)}(0) \overline{\xi^{-n} f(\xi)}/n! \right)^{-1} \right).$$

Equality holds for the last inequality if and only if  $f$  is of the form

$$f(q) = q^n \left( 1 - q(1 - qb\bar{\eta})^{-*} * (q\bar{\eta} - b) \frac{\overline{f^{(n)}(0)v}}{n!} \right)^{-*} * \left( \frac{f^{(n)}(0)}{n!} - q(1 - qb\bar{\eta})^{-*} * (q\bar{\eta} - b) \bar{v} \right),$$

where

$$b \in [-1, 1), \quad v = \left( \xi^n f^{(n)}(0)/n! - f(\xi) \right)^{-1} \xi \left( \xi^n - f(\xi) \overline{f^{(n)}(0)/n!} \right) \in \partial \mathbb{B},$$

and

$$\eta = \left( \xi^n - f(\xi) \overline{f^{(n)}(0)/n!} \right)^{-1} \xi \left( \xi^n - f(\xi) \overline{f^{(n)}(0)/n!} \right) \in \partial \mathbb{B}.$$

In particular,

$$\bar{\xi} \left( f(\xi) \overline{f'(\xi)} + [\bar{\xi}, f(\xi) \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}] \right) > n$$

unless  $f(q) = q^n u$  for some  $u \in \partial\mathbb{B}$ .

Moreover, it holds that

$$\left\langle f(t\xi), f(\xi) \right\rangle \geq t^n \frac{(\delta - n + 1)t - (\delta - n - 1)}{(\delta - n + 1) - (\delta - n - 1)t}, \quad \forall t \in (-1, 1),$$

with equality for some  $t_0 \in (-1, 1)$  if and only if

$$f(q) = q^n \left( q(\delta - n - 1) - \xi(\delta - n + 1) \right)^{-*} * \left( \xi(\delta - n - 1) - q(\delta - n + 1) \right) \bar{\xi}^n f(\xi).$$

*Proof of Theorem 6.6.* We first prove the assertion (i). In [37, Theorem 4], we have proved that

$$(6.8) \quad \frac{\partial |f|}{\partial \xi}(\xi) = \bar{\xi} \left( f(\xi) \overline{f'(\xi)} + [\bar{\xi}, f(\xi) \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}] \right).$$

So to obtain the desired sharp estimate in (6.3), it suffices to prove that

$$(6.9) \quad \frac{\partial |f|^2}{\partial \xi}(\xi) \geq \frac{4}{\mathcal{S} + \frac{1 - |f(0)|^2}{|f(\xi) - f(0)|^2}}$$

with  $\mathcal{S}$  being the same as in (6.4), we proceed as follows. Set

$$(6.10) \quad v = (f(0) - f(\xi))^{-1} \xi (1 - f(\xi) \overline{f(0)}),$$

which belongs to  $\partial\mathbb{B}$ , for  $f(\xi) \in \partial\mathbb{B}$  by assumption. Set

$$(6.11) \quad g(q) := (1 - f(q) \overline{f(0)})^{-*} * (f(0) - f(q))v,$$

then  $g$  is a slice regular function on  $\mathbb{B} \cup \mathbb{S}_{\xi}$  such that  $g(\mathbb{B}) \subseteq \mathbb{B}$ . Furthermore, it is evident that  $g(0) = 0$  and

$$(6.12) \quad g'(0) = -\frac{f'(0)}{1 - |f(0)|^2} v.$$

Denote

$$(6.13) \quad \eta = T_{1-f(0)*f^c}(\xi) \in \partial\mathbb{B},$$

which is a boundary fixed point of  $g$ . Indeed, it easily follows from Proposition 6.3, (6.10) and (6.11) that

$$(6.14) \quad g(\eta) = (1 - f(\xi) \overline{f(0)})^{-1} \xi (1 - f(\xi) \overline{f(0)}) = T_{1-f(0)*f^c}(\xi) = \eta,$$

and hence the slice regular function  $g$  satisfies all the assumptions in Theorem 6.6.

We next claim that

$$(6.15) \quad \frac{\partial |f|^2}{\partial \xi}(\xi) = \frac{|f(0) - f(\xi)|^2}{1 - |f(0)|^2} \lim_{t \rightarrow 0^+} \frac{1 - |g \circ T_{1-f(0)*f^c}(\xi - t\xi)|^2}{t}.$$

First, from (6.11) we obtain that

$$f(q) = (1 - g(q) \overline{v f(0)})^{-*} * (f(0) - g(q) \bar{v}).$$

This together with Proposition 6.3 implies

$$(6.16) \quad f(q) = \left( 1 - g \circ T_{1-g \overline{f(0)v}}(q) \overline{f(0)v} \right)^{-1} \left( f(0) - g \circ T_{1-g \overline{f(0)v}}(q) \bar{v} \right),$$

from which one easily deduces that

$$1 - |f(q)|^2 = \frac{(1 - |f(0)|^2)(1 - |g \circ T_{1-g \overline{f(0)v}}(q)|^2)}{|1 - g \circ T_{1-g \overline{f(0)v}}(q) \overline{f(0)v}|^2}.$$

Consequently,

$$\begin{aligned}
 (6.17) \quad \frac{\partial |f|^2}{\partial \xi}(\xi) &= \lim_{t \rightarrow 0^+} \frac{1 - |f(\xi - t\xi)|^2}{t} \\
 &= \frac{1 - |f(0)|^2}{|f(0) - g \circ T_{1-g\overline{f(0)v}}(\xi)\bar{v}|^2} \lim_{t \rightarrow 0^+} \frac{1 - |g \circ T_{1-g\overline{f(0)v}}(\xi - t\xi)|^2}{t}.
 \end{aligned}$$

Now a direct calculation gives that

$$1 - \overline{gf(0)v} = (1 - |f(0)|^2)(1 - \overline{ff(0)})^{-*},$$

which leads to

$$(6.18) \quad T_{1-g\overline{f(0)v}} = T_{(1-f\overline{f(0)})^{-*}} = T_{1-f(0)*f^c}.$$

This fact together with the notation of  $\eta$  in (6.13) implies that

$$(6.19) \quad \eta = T_{1-f(0)*f^c}(\xi) = T_{1-g\overline{f(0)v}}(\xi).$$

Furthermore, it follows from (6.16) and (6.19) that

$$g \circ T_{1-g\overline{f(0)v}}(\xi)\bar{v} = g(\eta)\bar{v} = \eta\bar{v} = (1 - f(\xi)\overline{f(0)})^{-1}\xi(f(0) - f(\xi))$$

and hence

$$(6.20) \quad |f(0) - g \circ T_{1-g\overline{f(0)v}}(\xi)\bar{v}| = \frac{1 - |f(0)|^2}{|f(0) - f(\xi)|}.$$

Now (6.15) immediately follows by substituting (6.18) and (6.20) into (6.17).

Next we turn to the estimate from below of the limit

$$\lim_{t \rightarrow 0^+} \frac{1 - |g \circ T_{1-f(0)*f^c}(\xi - t\xi)|^2}{t}$$

appeared in (6.15). At first sight, it should be the directional derivative of  $|g|^2$  along  $\eta$  at the boundary point  $\eta \in \partial\mathbb{B}$ . Unfortunately, it is in general not the case (It is obviously the case for  $\xi = 1$  or  $f(0) = 0$ ). Even though the smooth curve

$$t \mapsto \Gamma(t) := T_{1-g\overline{f(0)v}}(\xi - t\xi)$$

defined on some interval  $(-\varepsilon, \varepsilon)$  with  $\varepsilon > 0$  sufficiently small goes through the point

$$\Gamma(0) = T_{1-g\overline{f(0)v}}(\xi) = \eta \in \partial\mathbb{B},$$

its tangent vector  $\Gamma'(0)$  at  $t = 0$  is not necessarily the same as the direction  $\eta \in \partial\mathbb{B}$ . However, we still can estimate the above limit in virtue of Theorem 6.5. Indeed, applying Theorem 6.5 and Julia inequality (6.2) to the slice regular function  $h(q) := q^{-1}g(q)$  mapping  $\mathbb{B}$  to  $\mathbb{B}$  with  $h(\bar{\eta}) = 1$  yields that

$$\lim_{t \rightarrow 0^+} \frac{1 - |h \circ T_{1-f(0)*f^c}(\xi - t\xi)|^2}{t} \geq \frac{1}{\operatorname{Re}\left((1 - h(0))^{-1}(1 + h(0))\right)} = \frac{|1 - h(0)|^2}{1 - |h(0)|^2},$$

and hence

$$\begin{aligned}
 \lim_{t \rightarrow 0^+} \frac{1 - |g \circ T_{1-f(0)*f^c}(\xi - t\xi)|^2}{t} &= 1 + \lim_{t \rightarrow 0^+} \frac{1 - |h \circ T_{1-f(0)*f^c}(\xi - t\xi)|^2}{t} \\
 &\geq \frac{2(1 - \operatorname{Re} h(0))}{1 - |h(0)|^2} \\
 &\geq \frac{2}{1 + \operatorname{Re} h(0)} \\
 &= \frac{2}{1 + \operatorname{Re} g'(0)}.
 \end{aligned}
 \tag{6.21}$$

Now substituting (6.10), (6.12) and (6.21) into (6.15) yields the desired sharp estimate in (6.9), and thus completes the proof of inequality (6.3).

If equality holds for inequality in (6.3), then equalities hold for all the inequalities in (6.21), thus from the condition for equality in the Julia inequality (6.2) and the above deduction of (6.21) it follows that  $h$  is of the form

$$h(q) = (1 - qa\bar{\eta})^{-*} * (q\bar{\eta} - a) \tag{6.22}$$

with some constant  $a \in [-1, 1)$ . Consequently,  $f$  must be of the form

$$f(q) = \left(1 - q(1 - qa\bar{\eta})^{-*} * (q\bar{\eta} - a)\overline{f(0)v}\right)^{-*} * \left(f(0) - q(1 - qa\bar{\eta})^{-*} * (q\bar{\eta} - a)\bar{v}\right), \tag{6.23}$$

where  $a \in [-1, 1)$ , and  $v$  and  $\eta$  are the same as those in (6.10) and (6.13), respectively. Therefore, the equality in inequality (6.3) can hold only for slice regular self-mappings of the form (6.23), and a direct calculation shows that it does indeed hold for all such slice regular self-mappings. Now to complete the proof of (i), it remains to prove inequality (6.6). To this end, we use the splitting lemma (cf. [14, Lemma 1.3]). Let  $I \in \mathbb{S}$  be such that  $\xi \in \partial\mathbb{B} \cap \mathbb{C}_I$  and let us split the slice regular function  $f\overline{f(\xi)}$  as

$$f(z)\overline{f(\xi)} = \varphi(z) + \psi(z)J, \quad \forall z \in \mathbb{B}_I,$$

where  $J \in \mathbb{S}$  and  $J \perp I$ , and  $\varphi, \psi$  are two holomorphic self-mappings of  $\mathbb{B}_I$  satisfying

$$|f(z)|^2 = |\varphi(z)|^2 + |\psi(z)|^2 \tag{6.24}$$

for all  $z \in \mathbb{B}_I$ . Moreover, it is evident that

$$\varphi(\xi) = 1, \quad \psi(\xi) = 0,$$

and

$$\langle f(t\xi), f(\xi) \rangle = \operatorname{Re}(f(t\xi)\overline{f(\xi)}) = \operatorname{Re} \varphi(t\xi).$$

Now inequality (6.6) follows immediately by applying Minda's theorem (see [30, p. 135, Theorem 1]) to the holomorphic self-mapping  $\varphi$  of  $\mathbb{B}_I$  and noticing that

$$\delta = \frac{\partial|f|}{\partial\xi}(\xi) = \frac{\partial|\varphi|}{\partial\xi}(\xi) = \xi\varphi'(\xi).$$

Here the last equality follows directly from an elementary geometric consideration about  $\varphi$  at the boundary point  $\xi$  or alternatively from the classical Julia-Wolff-Carathéodory theorem (cf. [39]; also [40, p. 48 (VI-3)]).

If equality holds for inequality (6.6) at some  $t_0 \in (-1, 1)$ , then it again follows from Minda's theorem that

$$\varphi(z) = \frac{(\delta - 1)\xi - (\delta + 1)z}{(\delta - 1)z - (\delta + 1)\xi}, \quad \forall z \in \mathbb{B}_I. \tag{6.25}$$

Furthermore, it follows from equality in (6.24) that

$$|\psi(z)|^2 = |f(z)|^2 - |\varphi(z)|^2 \leq 1 - |\varphi(z)|^2, \quad \forall z \in \mathbb{B}_I,$$

which together with (6.25) implies that  $\psi \equiv 0$ , in virtue of the maximum principle, and hence  $f$  must be of the form in (6.7). This completes the proof of (i) and it remains to prove (ii).

However, (ii) follows easily from (i) by considering the slice regular function  $h(q) := q^{-n}f(q)$  and noticing that

$$h(0) = \frac{f^{(n)}(0)}{n!}, \quad h'(0) = \frac{f^{(n+1)}(0)}{(n+1)!}.$$

Moreover,

$$f(\xi)\overline{f'(\xi)} + [\bar{\xi}, f(\xi)\overline{R_{\bar{\xi}}R_{\xi}f(\xi)}] = n\xi + h(\xi)\overline{h'(\xi)} + [\bar{\xi}, h(\xi)\overline{R_{\bar{\xi}}R_{\xi}h(\xi)}]$$

as one easily verifies. Now the proof is complete.  $\square$

*Remark 6.7.* In the preceding proof of the desired sharp estimate in (6.3), the second inequality in (6.21) plays a key role. If we make use of the first one in (6.21), we will obtain more precise estimate than that in (6.3). Formally, this estimate will be much more complicated, but it is the same as that in (6.3) if the functions of concern are the extremal functions given in (6.5).

*Remark 6.8.* From inequality (6.6), we can obtain the following estimate:

$$(6.26) \quad \langle \xi^2 f''(\xi), f(\xi) \rangle \geq \delta(\delta - 1).$$

Indeed, from inequality (6.6) and the notion of  $\delta$  it follows that

$$\begin{aligned} \langle f(t\xi) - f(\xi) - (t-1)\xi f'(\xi), f(\xi) \rangle &= \langle f(t\xi), f(\xi) \rangle - 1 - (t-1)\delta \\ &\geq \frac{(\delta+1)t - (\delta-1)}{(\delta+1) - (\delta-1)t} - 1 - (t-1)\delta \\ &= (t-1)^2 \frac{\delta(\delta-1)}{(\delta+1) - (\delta-1)t} \end{aligned}$$

for all  $t \in (-1, 1)$ . Now dividing by  $(t-1)^2$  on both sides and then letting  $t \rightarrow 1^-$  yields (6.26). Alternatively, (6.26) can also be proved by an argument by means of the convexity of  $f(\mathbb{B})$  at the point  $\xi \in \partial\mathbb{B}$ . This argument seems more natural in principle, but rather difficult to deal with in practise, because of the computation of the second order differential of  $f$ .

Conversely, inequality (6.26) reveals in a certain sense the convexity of the image  $f(\mathbb{B})$  at the point  $\xi \in \partial\mathbb{B}$ . For simplicity, we further assume that the slice regular function  $f$  in Theorem 6.6 maps  $\mathbb{B}_I$  into itself, i.e.  $f(\mathbb{B}_I) \subseteq \mathbb{B}_I$ , where  $I = I_{\xi}$  is the pure imaginary unit identified by  $\xi \in \partial\mathbb{B}$ . In this special case,  $\delta$  is precisely the positive number  $\xi f'(\xi)/f(\xi)$ , and inequality (6.26) becomes

$$\operatorname{Re}\left(\frac{\xi^2 f''(\xi)}{f(\xi)}\right) \geq \delta(\delta - 1).$$

We then obtain that

$$(6.27) \quad \operatorname{Re}\left(\frac{\xi f''(\xi)}{f'(\xi)} + 1\right) \geq \delta > 0,$$

which together with a well-known analytical characterization of convexity (cf. [26, Theorem 2.2.3]) implies that  $f(\mathbb{B}_I)$  is convex at  $\xi \in \partial\mathbb{B}$ . Furthermore, what is more interesting

is that as shown by Theorem 6.6 (ii) and inequality (6.27), the higher the vanishing order of  $f$  at the origin 0 is, the more convex at the boundary point  $\xi \in \partial\mathbb{B}$  the image  $f(\mathbb{B}_I)$  of  $\mathbb{B}_I$  under  $f$  is, i.e. the bigger the number

$$\operatorname{Re}\left(\frac{\xi f''(\xi)}{f'(\xi)} + 1\right)$$

is. Intuitively, this is indeed the case.

**6.3. Some corollaries of Theorem 6.6.** First notice that the term on the right-hand side of inequality (6.3) is clearly positive, for

$$|f'(0)| \leq 1 - |f(0)|^2$$

as shown by the Schwarz-Pick lemma (see [1, 3]). Replacing the real part in the notation of  $\mathcal{S}$  appearing in inequality (6.3) by modulus yields inequality (6.28) below. Hence, the following corollary is a weaker version of Theorem 6.6.

**Corollary 6.9.** *Let  $\xi \in \partial\mathbb{B}$  and  $f$  be a slice regular function on  $\mathbb{B} \cup \mathbb{S}_\xi$  such that  $f(\mathbb{B}) \subseteq \mathbb{B}$  and  $f(\xi) \in \partial\mathbb{B}$ . Then*

(i) *the following sharp estimate holds:*

$$(6.28) \quad \bar{\xi}\left(f(\xi)\overline{f'(\xi)} + [\bar{\xi}, f(\xi)\overline{R_{\bar{\xi}}R_{\xi}f(\xi)}]\right) \geq \frac{2|f(\xi) - f(0)|^2}{1 - |f(0)|^2 + |f'(0)|}.$$

*Moreover, equality holds for the last inequality if and only if  $f$  is of the form*

$$(6.29) \quad f(q) = \left(1 - q(1 - qa\bar{\eta})^{-*} * (q\bar{\eta} - a)\overline{f(0)v}\right)^{-*} * \left(f(0) - q(1 - qa\bar{\eta})^{-*} * (q\bar{\eta} - a)\bar{v}\right),$$

*where*

$$a \in [-1, 0], \quad v = (f(0) - f(\xi))^{-1}\xi(1 - f(\xi)\overline{f(0)}) \in \partial\mathbb{B},$$

*and*

$$\eta = (1 - f(\xi)\overline{f(0)})^{-1}\xi(1 - f(\xi)\overline{f(0)}) \in \partial\mathbb{B}.$$

(ii) *if further*

$$f^{(k)}(0) = 0, \quad \forall k = 0, 1, \dots, n-1$$

*for some  $n \in \mathbb{N}$ , then*

$$\bar{\xi}\left(f(\xi)\overline{f'(\xi)} + [\bar{\xi}, f(\xi)\overline{R_{\bar{\xi}}R_{\xi}f(\xi)}]\right) \geq n + \frac{2|f(\xi) - \xi^n f^{(n)}(0)/n!|^2}{1 - |f^{(n)}(0)/n!|^2 + |f^{(n+1)}(0)/(n+1)!}.$$

*Moreover, equality holds for the last inequality if and only if  $f$  is of the form*

$$f(q) = q^n \left(1 - q(1 - qb\bar{\eta})^{-*} * (q\bar{\eta} - b)\frac{\overline{f^{(n)}(0)v}}{n!}\right)^{-*} * \left(\frac{f^{(n)}(0)}{n!} - q(1 - qb\bar{\eta})^{-*} * (q\bar{\eta} - b)\bar{v}\right),$$

*where*

$$b \in [-1, 0], \quad v = \left(\xi^n f^{(n)}(0)/n! - f(\xi)\right)^{-1}\xi\left(\xi^n - f(\xi)\overline{f^{(n)}(0)/n!}\right) \in \partial\mathbb{B},$$

*and*

$$\eta = \left(\xi^n - f(\xi)\overline{f^{(n)}(0)/n!}\right)^{-1}\xi\left(\xi^n - f(\xi)\overline{f^{(n)}(0)/n!}\right) \in \partial\mathbb{B}.$$

*Proof.* We only give a proof of the assertion (i), the other one being similar. Inequality (6.28) follows immediately by replacing the real part in the notation of  $\mathcal{S}$  appearing in inequality (6.3) by modulus, and equality in (6.28) holds if and only if

$$f'(0)(f(\xi) - f(0))^{-1}\xi(1 - f(0)\overline{f(\xi)})^{-1} \in \mathbb{R}^+,$$

which is equivalent to  $h(0) \in [0, 1]$ , i.e.  $a \in [-1, 0]$ . Here the function  $h$  is the one in (6.22).  $\square$

Clearly, Corollary 6.9 implies [37, Theorem 4]:

$$\bar{\xi} \left( f(\xi) \overline{f'(\xi)} + [\bar{\xi}, f(\xi) \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}] \right) \geq \frac{2(1 - |f(0)|)^2}{1 - |f(0)|^2 + |f'(0)|^2},$$

and provides additionally all the extremal functions:

$$f(q) = \left( 1 + q(1 - qa\bar{\xi})^{-*} * (q\bar{\xi} - a)\bar{\xi}|f(0)| \right)^{-*} * \left( |f(0)| + q(1 - qa\bar{\xi})^{-*} * (q\bar{\xi} - a)\bar{\xi} \right) f(\xi)$$

with  $a \in [-1, 0]$ . If the slice regular function  $f$  considered in Theorem 6.6 has the interior fixed point 0 and a boundary fixed point  $\xi \in \partial\mathbb{B}$ , then Theorem 6.6 (i) implies:

**Corollary 6.10.** *Let  $\xi \in \partial\mathbb{B}$  and  $f$  be a slice regular function on  $\mathbb{B} \cup \mathbb{S}_{\xi}$  such that  $f(\mathbb{B}) \subseteq \mathbb{B}$ ,  $f(0) = 0$  and  $f(\xi) = \xi$ . Then*

$$f'(\xi) - [\xi, R_{\bar{\xi}} R_{\xi} f(\xi)] \geq \frac{2}{1 + \operatorname{Re} f'(0)}.$$

Moreover, equality holds for the last inequality if and only if  $f$  is of the form

$$f(q) = q(1 - qa\bar{\xi})^{-*} * (q - a\xi)\bar{\xi}$$

for some constant  $a \in [-1, 1]$ .

As indicated in [37, Example 2], the Lie bracket in the preceding corollary does not vanish and thus  $f'(\xi)$  is not necessarily a positive real number, in general. However, the same line of the proof of Theorem 6.6 implies simultaneously the following theorem, which provides a sharp lower bound for  $|f'(\xi)|$ .

**Theorem 6.11.** *Let  $\xi \in \partial\mathbb{B}$  and  $f$  be a slice regular function on  $\mathbb{B} \cup \{\xi\}$  such that  $f(\mathbb{B}) \subseteq \mathbb{B}$ ,  $f(0) = 0$  and  $f(\xi) = \xi$ . Then*

$$|f'(\xi)| \geq \frac{2}{1 + \operatorname{Re} f'(0)}.$$

Moreover, equality holds for the last inequality if and only if  $f$  is of the form

$$f(q) = q(1 - qa\bar{\xi})^{-*} * (q - a\xi)\bar{\xi}$$

for some constant  $a \in [-1, 1]$ .

We now conclude this paper with a comparison of the results proved in this section and the corresponding results for holomorphic self-mappings of the open unit disc on the complex plane. Even in the complex setting, the result obtained in Theorem 6.6 is a new result. More precisely, for every holomorphic function  $f$  on  $\mathbb{D} \cup \{1\}$  (Since the automorphism group of biholomorphisms of the open unit disk  $\mathbb{D} \subset \mathbb{C}$  acts bi-transitively on the boundary  $\partial\mathbb{D}$ , we can assume without loss of generality that the boundary point  $\xi \in \partial\mathbb{D}$  under consideration is 1) satisfying that  $f(\mathbb{D}) \subseteq \mathbb{D}$  and  $f(1) = 1$ , it can extend regularly and uniquely to  $\mathbb{B} \cup \{1\}$ . We denote (with a slight abuse of notation) this unique regular extension still by  $f$  itself. Thus  $f$  is a slice regular function on  $\mathbb{B} \cup \{1\}$

such that  $f(\mathbb{B}) \subseteq \mathbb{B}$  and  $f(1) = 1$ . The assertion that  $f(\mathbb{B}) \subseteq \mathbb{B}$  follows easily from a convex combination identity in [36]. For all such  $f$ , our result becomes

$$(6.30) \quad f'(1) \geq \frac{2}{\operatorname{Re}\left(\frac{1 - f(0)^2 + f'(0)}{(1 - f(0))^2}\right)},$$

which implies

$$f'(1) \geq \frac{2|1 - f(0)|^2}{1 - |f(0)|^2 + |f'(0)|}.$$

These two inequalities improve the following estimate (also called Osserman's inequality) established by Osserman in [32]:

$$f'(1) \geq \frac{2(1 - |f(0)|)^2}{1 - |f(0)|^2 + |f'(0)|}.$$

This new estimate in (6.30) for holomorphic self-mappings of the open unit disk  $\mathbb{D}$ , with boundary regular fixed point 1, was initially proved in [10, Theorem 3] via an analytic semigroup approach and Julia-Wolff-Carathéodory theorem for univalent holomorphic self-mappings of  $\mathbb{D}$ , which was derived by the method of extremal length. The method presented in [10] can not be used to get the extremal functions for which equality holds in (6.30). The proof presented in this paper for the special case that  $\xi = 1$  is quite elementary, and has its extra advantage of getting the extremal functions.

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